

Summer 2020 High School Math Camp: Pigeonhole Principle

Sam Auyeung

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The pigeonhole principle is a very simple idea: if I have $k + 1$ objects and k boxes, then if I want to put all the objects into the boxes, at least one box will end up with more than one object.

For some reason, this idea was presented with pigeons and pigeonholes and the name stuck. But I've also heard it called the box principle.

For example, if we have 5 students and 4 types of sandwiches, then at least two students will eat the same type of sandwich (assuming they do eat and all).

Note that we can generalize this principle. Say we have 10 objects and 3 boxes. Then convince yourself that at least one box has 4 objects in it. Here is a more rigorous statement: If you have n objects and b boxes, then at least one box has $\lceil n/b \rceil$ objects. The symbols here mean you take n/b then next largest integer.

Warmup:

1. Among 13 people, at least two are born in the same month. How many people do you need to guarantee that at least two are born on the same calendar day?
2. Let a, b, c, d, e be any integers. Then at least three of them are all even or all odd.
3. Take any $n + 1$ integers. Show that there are least two whose difference is a multiple of n .
4. People are seated around a circular table at a restaurant. The food is placed on a circular platform in the center of the table, and this circular platform can rotate (this is commonly found in Chinese restaurants that specialize in banquets). Each person ordered a different entrée, and it turns out that no one has the correct entrée in front of them. Show that it is possible to rotate the platform so that at least two people will have the correct entrée.
5. Take the xy plane and color every point red or blue. Show that there is always a pair of points of the same color so that the two points are exactly 1 mile apart.
6. Take 5 points and place them in a square of side length 1. Show that there are two of them that are no more than $1/\sqrt{2}$ units apart.
7. Given any five points on a sphere, prove that there is a closed hemisphere that contains four of them.

The following puzzles all can be solved using the pigeonhole principle (possibly applied more than once) and sometimes a bit more knowledge of math. But no fancy mathematical machinery is needed.

Puzzles:

1. Show that on Facebook (or any social media platform where the notion of “friends” exists), there are two people with the same number of friends (not necessarily the same friends). Remember that if you’re friends with someone, they are also friends with you.
2. Show that the decimal expansion of a rational number is periodic.
3. Consider all the integer points in the xy -plane (this forms a grid or lattice). For example, $(-5, 12)$. Suppose we can color each point red, green, or blue any way we like. Show that for any coloring scheme, there are always four points of the same color which form the vertices of a rectangle. The coloring schemes can be very random.

Can you show a similar result when there are k possible colors?

4. Choose any 10 distinct 2-digit numbers. So we can choose between 10 and 99. Show that there are two disjoint (nonempty) subsets with the same sum when you add up the elements.

If you’re not familiar with the terms here: Say we have the numbers 10, 11, 13, 29, 32, 54, 61, 73, 83, 92. A subset is any collection of these. For example, $\{11, 61, 83\}$ is a subset. $\{11\}$ is a subset. Having nothing is also a subset but we call that empty $\{\}$. We can even take the whole thing as a subset.

Two subsets are disjoint if they have no elements in common. So $A = \{10, 11, 29\}$ and $B = \{13, 54, 73, 99\}$ are disjoint. But B is not disjoint from $\{11, 54, 73, 83\}$.

5. Let a be a positive integer that is not divisible by 2 or 5. Show that for any positive integer n , there exists a power of a such that a ends with (in the decimal system)

$$\underbrace{00\dots01}_n.$$

6. Place 41 rooks on a 10×10 chessboard. We’ll say that a rook is unable to attack another rook if they are in different rows and different columns. Show that there are 5 rooks that are not attacking each other. We’ll ignore the fact that in regular chess, rooks cannot attack a piece of there is a third piece in between.

7. Groups of people:

- (a) Consider a group of 6 people. Then there are either three people who all mutually know each other or three who are mutually complete strangers.
- (b) Consider a group of 17 scientists who correspond with everyone else in the group on three scientific subjects, say rocket science, geology, and biology. For any pair of scientists, they talk to each other only about one topic but of course, they can talk to other people about different topics. Show that at least three of them correspond with each other about the same subject.

8. Let A and B be 2×2 matrices with integer coefficients. Suppose that $A, A + B, A + 2B, A + 3B, A + 4B$ are all invertible. Show that $A + 5B$ is also invertible.
9. Let S be any region in the plane with area greater than some positive integer n . Show that we can translate S around so that it covers $n + 1$ lattice points (points with integer coordinates like $(-3007, 104)$).