Virtual Fundamental Chains and Cycle Techniques

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These are notes taken from an introduction by John Morgan who was the editor for *Virtual Fundamental Cycles in Symplectic Topology*. The book comes from a 2014 workshop at the Simons Center for Geometry and Physics, titled *Moduli Spaces of Pseudo-holomorphic curves and their applications to Symplectic Topology*. The contributors to this book are Dusa McDuff, Mohammad Tehrani, Kenji Fukaya, and Dominic Joyce.

1 Introduction

Let (M^{2n}, ω) be a symplectic manifold with $\mathcal{J}_\omega$ being the space of compatible almost complex structures. Recall that this space is contractible. Then if we consider the moduli space of J_0-holomorphic curves of genus g with k marked points (which is equivalent to considering a quotient by automorphisms), we can ask what happens if we change J_0. That is, what happens to $\mathcal{M}_{g,k}(J_0)$ if we replaced this with a path $J(t) \in \mathcal{J}_\omega$ connecting J_0 and J_1. I think that if the path passes through only regular ACS, we then get a cobordism W with $\partial W = \mathcal{M}_{g,k}(J_0) \sqcup \mathcal{M}_{g,k}(J_1)$. We might not always get a cobordism in this manner though when $J(t)$ isn't a regular path. On the other hand, the cobordism class of $\mathcal{M}_{g,k}(J_0)$ is an invariant of (M, ω).

Now, the dimension of these moduli spaces can be computed with the Riemann-Roch theorem or more generally, Atiyah-Singer index theorem. However, to apply these theorems, we need transversality. When we make additional assumptions such as monotonicity or semi-positivity, then we are guaranteed transversality for generic J. Without these assumptions, we have obstructions to transversality arising from ramified multiple covers of transversal solutions. The formal dimensions of these covers can be **negative** whilst the spaces themselves are **nonempty**. Thus, virtual techniques were introduced to handle such issues.

2 Example

2.1 Transversality

Let (X^{2n}, ω) be a symplectic manifold. Let $\mathcal{M}_{0,3}$ be the space of all maps $u : S^2 \to X$ with three marked points: $0, 1, \infty$. We fix a 2-cycle α; we can further require that the maps u satisfy $u_*[S^2] = \alpha$. Call this space $\mathcal{M}_{0,3}(\alpha)$. We'll assume that

$$\langle c_1(TX), \alpha \rangle = \int_{[\alpha]} c_1 = \int_{S^2} u^* c_1 > 0.$$

Then the dimension is $\dim \mathcal{M}_{0,3}(\alpha) = 2\langle c_1, \alpha \rangle + 2n$. There is also an evaluation map $ev : \mathcal{M}_{0,3}(\alpha) \to X \times X \times X$ which sends $u \mapsto (u(0), u(1), u(\infty))$. We choose cycles $Q_1, Q_2, Q_3 \subset X$

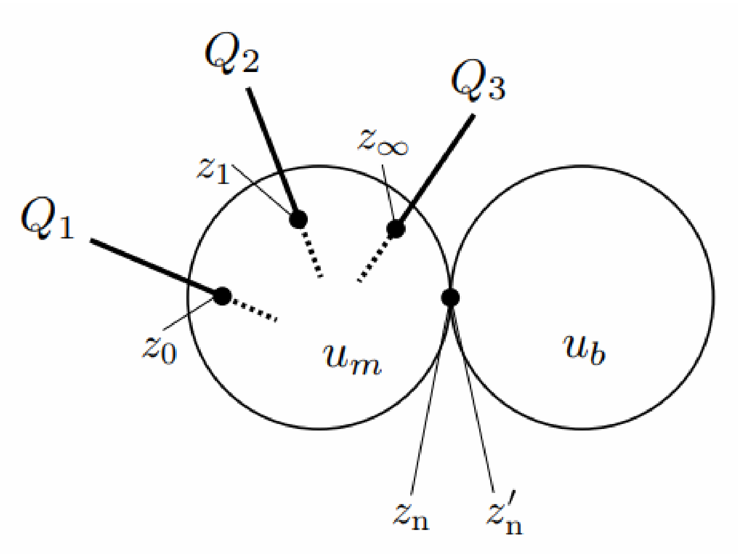
such that

$$\sum_{i=1}^{3}(2n - \dim Q_i) = 2\langle c_1, \alpha\rangle + 2n.$$

Then, the formal dimension of the moduli space $\mathcal{M}_{0,3}(\alpha; Q_1, Q_2, Q_3) := \{u \in \mathcal{M}_{0,3}(\alpha) : (u(0), u(1), u(\infty)) \in Q_1 \times Q_2 \times Q_3\}$ is zero. For generic $J \in \mathcal{J}_\omega$, the assumption $\langle c_1, \alpha\rangle > 0$ gives us transversality.

2.2 Compactness

However, $\mathcal{M}_{0,3}(\alpha; Q_1, Q_2, Q_3)$ is not necessarily compact. For example, we can have a sequence $\{u_i\} \in \mathcal{M}_{0,3}(\alpha; Q_1, Q_2, Q_3)$ which has a subsequence limiting to a union of pair of maps. This is Gromov's compactness theorem about stable maps.



Here in the picture, we have that $\lim u_i(0) = z_0 \in Q_1$ and similarly for the other two points. The $u_m : S^2 \to X$ is a map with four marked points; there's this extra point z_n. $u_b : S^2 \to X$ is a map with one marked point: z'_n and it's identified with z_n. Let $\alpha_m = (u_m)_*[S^2]$ and $\alpha_b = (u_b)_*[S^2]$. Then, $\alpha = \alpha_m + \alpha_b$. Observe that $(u_m; z_0, z_1, z_\infty, z_n) \in \mathcal{M}_{0,4}(\alpha_m; Q_1, Q_2, Q_3)$ and $(u_b, z'_n) \in \mathcal{M}_{0,1}(\alpha_b)$. The virtual dimensions of the moduli spaces are given by

$$\dim \mathcal{M}_{0,4}(\alpha_m; Q_1, Q_2, Q_3) = 2\langle c_1, \alpha_m\rangle + 2n + 2 - \sum_{i=1}^{3}(2n - \dim Q_i)$$

$$\dim \mathcal{M}_{0,1}(\alpha_b) = 2\langle c_1, \alpha_b\rangle + 2n - 4.$$

How shall we explain this +2 and -4 in the dimension formulae? $(S^2, z_0, z_1, z_\infty, z_n)$ of u_m has a 2-dim moduli space while (S^2, z'_n) of u_b has an automorphism group of dim 4. I think we just want holomorphic maps $f : S^2 = \mathbb{C} \cup \{\infty\} \to S^2$ which are just rational maps. But we also want them to be injective so that leaves maps of the form $f(z) = az + b$ with $a, b \in \mathbb{C}$. Hence, dim = 4. We now observe that:

$$\begin{aligned}
&\dim \mathcal{M}_{0,4}(\alpha_m; Q_1, Q_2, Q_3) + \dim \mathcal{M}_{0,1}(\alpha_b) - 2n \\
=&2\langle c_1, \alpha_m\rangle + 2n + 2 - \sum_{i=1}^{3}(2n - \dim Q_i) + 2\langle c_1, \alpha_b\rangle + 2n - 4 - 2n.
\end{aligned}$$

Since $\alpha = \alpha_m + \alpha_b$ and we have this condition on our Q_i's, we find that the sum of the two dimensions minus $2n$ is, in the end, -2. This means that **if** we have transversality, then these configurations do not appear. Thus, to obtain compactness, we'll like the limiting moduli to be cut ou tranversally and therefore, be empty.

Unfortunately, $\mathcal{M}_{0,1}(\alpha_b)$ is not transversal for generic almost complex structure. Consider when $\alpha_b = k\alpha'_b$ with $2\langle c_1, \alpha'_b\rangle < 0$. Then, there can be a J-holomorphic curve with $2\langle c_1, \alpha'_b\rangle + 2n - 4 \geq 0$ but $2\langle c_1, \alpha_b = k\alpha'_b\rangle + 2n - 4 < 0$. For example, this occurs when $n = 3, k = 2$ and $\langle c_1, \alpha'_b\rangle = -1$. Then the first is 0, the second is -2.

Because the first inequality allows says the dimension is non-negative, we cannot conclude that the moduli space $\mathcal{M}_{0,1}(\alpha'_b)$ is empty via a dimension count approach. In fact, it could be nonempty for all ACS. Moreover, if $u' \in \mathcal{M}_{0,1}(\alpha'_b)$, then we will have $u(z) := u'(z^k) \in \mathcal{M}_{0,1}(\alpha_b)$; so $\mathcal{M}_{0,1}(\alpha_b) \neq \varnothing$. So this space $\mathcal{M}_{0,1}(\alpha_b)$ has **negative** virtual dimension but is also **nonempty** for **all** almost complex structures. It cannot be cut out transversally.

3 Virtual Techniques

The conclusion is that we have no hope of perturbing the ACS to get transversality in order to obtain a fundamental class of the moduli space, when there are such multiple covers of formally negative dimension. So we need some virtual fundamental chain or cycle techniques.

The advantage to this is that we can resolve the issue in almost every situation. The disadvantage is that we now use $\mathbb{Q}$ instead of $\mathbb{Z}$. This means that the Gromov-Witten invariants are fractions. But this isn't so novel; the Euler number of an orbifold is generally in $\mathbb{Q}$. Recall, an orbifold is locally like U/Γ where $U = B(r, 0) \subset \mathbb{R}^n$ and Γ is a finite group which acts orthogonally (so is an isometry).

Example 3.1. Let X be an orbifold defined as the union of $\mathbb{C}/\mathbb{Z}_2$ and $\mathbb{C}/\mathbb{Z}_3$. We identify $[z^2] \in \mathbb{C}/\mathbb{Z}_2$ and $[w^3] = [1/z^2] \in \mathbb{C}/\mathbb{Z}_3$. X has two singular points: $z = 0 \in \mathbb{C}/\mathbb{Z}_2$ and $w = 0 \in \mathbb{C}/\mathbb{Z}_3$.

There exists a vector field vanishing only at z and w; due to the non-trivial isotopy, we count these with multiplicity 1/2 and 1/3. So then the Euler number is $1/2 + 1/3 = 5/6$.

3.1 Techniques from *Virtual Fundamental Cycles in Symplectic Topology*

The versions of virtual techniques that McDuff, Fukaya, and Joyce use have some differences but fundamentally, follow three steps.

1. Represent a moduli space locally as a zero set of certain section s of an orbibundle $E \to U$ where U is an orbifold.
2. Regard such a local description of the moduli space as a "coordinate chart" of a certain geometric object and introduce an appropriate notion of "coordinate change."
3. Do one of the following
   (a) Either "Perturb" the section s in a way compatible with the coordinate changes so that it becomes transversal to the zero section and then glue the zero sets of perturbed sections in "various coordinate charts" by the coordinate change (McDuff and Fukaya's approach);
   (b) Or develop and use general techniques from "derived topology" to produce a fundamental chain or cycle from the geometric object in Step 2 (Joyce's approach).

It is worth pointing out that the arguments for Step 3 use only the formal properties of the geometric objects constructed in Step 2.

3.2 Another View

Another process I was told about by Mark McLean comes from a paper by B. Siebert (1996): *Gromov-Witten invariants of general symplectic manifolds.*

If we consider stable nodal J-holomorphic curves into a symplectic manifold (M, ω) of genus g with k marked points representing a homology class $\beta \in H_2(M, \mathbb{Z})$ up to automorphisms, we obtain a moduli space, usually denoted $\mathcal{M}_{g,k}(J, \beta)$. This space is compact under Gromov convergence. Though it is an orbifold because we can have multiply covered curves, we can still give it a fundamental class.

Let $m = 2(c_1(\beta) + (n-3)(1-g) + k)$ which is the virtual dimension of $\mathcal{M}_{g,k}(J, \beta)$. The claim is that there is a class called $[\mathcal{M}]^{\text{vir}}$ in the dual of Čech cohomology: $\check{H}^m(\mathcal{M}, \mathbb{Q})^*$. Apparently, the proper way to consider this is via strong homology but not many people take this view.

Anyways, how do we construct the virtual fundamental class? We make use of a theorem by Siebert:

Theorem 3.2. *(Siebert, '96) There exists a "natural" topological orbibundle $\pi : E \to B$ where B is an oriented space, with a continuous section s such that $\mathcal{M}_{g,k}(J, \beta) = s^{-1}(0)$. The bundle is unique up to some stabilization operation.*

The idea is that this is some finite dimensional approximation of the section $\bar{\partial}_J$ of some Banach bundle. Moreover, $2 \dim M - \dim E = m$. The construction is then to take neighborhoods $U_i \subset E$ of $s^{-1}(0)$ such that $\bigcap^\infty U_i = s^{-1}(0)$. Let $A_i \in \check{H}^m(U_i, \mathbb{Q})^*$ be the intersection product of the Borel-Moore homology of $[B \cap U_i]$ with $[s(B) \cap U_i]$. Here, we're thinking of B as the zero section of E in the first class. Then the virtual fundamental class is

$$[\mathcal{M}_{g,k}(J, \beta)] = \varinjlim_{i \in \mathbb{N}} A_i \in \varinjlim_{i \in \mathbb{N}} \check{H}^m(U_i, \mathbb{Q})^* \cong \check{H}^m(s^{-1}(0), \mathbb{Q})^*.$$

To reiterate, the basic idea is to use these U_i to approximate $s^{-1}(0)$, which by Siebert's theorem, is $\mathcal{M}_{g,k}(J, \beta)$.