

Introduction to Vector Calculus

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Caveat lector: These notes are mainly for myself as an instructor so some of the content might be beyond the scope of what I actually teach in class.

A **vector field** $V : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an assignment of a vector to every point of $\mathbb{R}^n$ (or a subset of it). Since there are global coordinates on $\mathbb{R}^n$, one can view a vector field as an n -tuple of multivariable functions: $V(x_1, \dots, x_n) = \langle F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n) \rangle$. We say that V is continuous, differentiable, smooth, etc. if each of the functions F_i are continuous, differentiable, smooth, etc. To be more precise, a vector field is really assigning to a point $p \in \mathbb{R}^n$ a vector $v \in T_p \mathbb{R}^n$, the tangent space of $\mathbb{R}^n$ at p . However, since $\mathbb{R}^n$ is a vector space already, we typically identify the whole space with each of its tangent spaces. Though all the math here makes sense in its own right, vector fields show up in many settings, primarily in physics. For example, a vector field can represent the movement of fluids like air or water, the flow of heat or electric and magnetic fields. If you've ever watched the news, chances are, you've seen animations of vector fields showing the movement of air/clouds, pressure gradients, or thermalclines.

We can also restrict V to subsets of $\mathbb{R}^n$; some of these subsets are themselves smooth submanifolds such as the unit sphere $S^2 \subset \mathbb{R}^3$. It's not necessarily the case that when we restrict V to S^2 or any other submanifolds of $\mathbb{R}^n$ that V lies in the tangent spaces of S^2 or those submanifolds. Indeed, there is an outward pointing unit normal vector field on S^2 ; the vectors are perpendicular to the tangent planes of S^2 rather than living inside of them.

It turns out that there is a strong relationship between vector fields and topology so it is convenient at this time to intuitively introduce the concept of **simply connectedness**. A space is simply connected if every loop in the space contracts to a point. Examples include $\mathbb{R}^n$ and the unit sphere $S^{n-1} \subset \mathbb{R}^n$ for $n \geq 2$. Nonexamples include an annulus in $\mathbb{R}^2$ or the n -torus T^n .

Lastly, it will be useful to consider special vector fields called **conservative** vector fields. They are simply those which are the gradient of a function. So let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$; we typically want the function to at least have continuous 2nd partial derivatives. Then $\nabla \phi = \langle \partial_{x_1} \phi, \dots, \partial_{x_n} \phi \rangle$. The function ϕ is called the **potential function**; it is not unique since we can just add a constant to it but it is very useful as we'll later see. The nomenclature comes from the fact that a certain quantity is always conserved even as some other data changes. In physics, that quantity is the net energy (work) and this is why ϕ is called the potential function since it describes potential energy in certain contexts.

1 Line Integration

Let $r : [a, b] \rightarrow \mathbb{R}^n$ be a parametrization of the geometric curve C . Then the **line integral** of a vector field V along C is denoted by

$$\int_C V \cdot dr := \int_a^b V(r(t)) \cdot r'(t) dt.$$

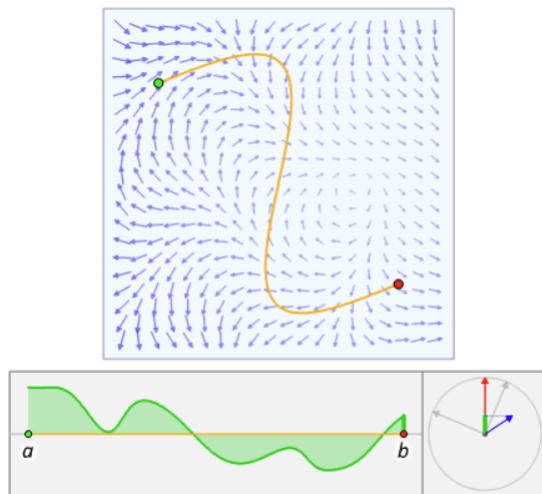
Here, we're precomposing the vector field by r in order to get a vector-valued function $V(r(t)) : [a, b] \rightarrow \mathbb{R}^n$ and then taking the dot product of that with another vector-valued function, the derivative $r'(t)$. The result is a scalar-valued function $[a, b] \rightarrow \mathbb{R}$ and hence, it is just a single variable function that we can integrate using techniques of Calculus I and II. Some remarks are in order.

1. This is called a line integral even though we're looking at a curve in space because the map r sends a line segment to some curve which may wind, twist, and intersect itself.
2. We didn't actually need to use a vector field; we could simply have integrated some function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ by taking $\int_a^b f(r(t)) dt$.
3. The parametrization $r(t)$ automatically gives us an orientation on C . If we use $s(t) := r(a+b-t)$ instead, then this is a parametrization that travels "backwards" along C ; note that $s'(t) = -r'(a+b-t)$ and the minus sign would change the sign of the line integral.
4. The reason for the notation $\int_C V \cdot dr$ and in particular, to emphasize the geometric curve C , is that the line integral is actually independent of the parametrization. That is, if $s : [c, d] \rightarrow \mathbb{R}^n$ is any other parametrization of C (not the one from above) where both s and r do not have vanishing derivatives, then there exists a function $\psi : [a, b] \rightarrow [c, d]$ (a diffeomorphism in fact) such that $r \circ \psi = s$. Then $\frac{d}{dt}(r \circ \psi) = dr \circ \psi'$ and this shows that

$$\int_a^b V(r(t)) \cdot r'(t) dt = \int_c^d V(s(t)) \cdot s'(t) dt.$$

5. In physics, we might consider C as describing the path of a moving charged particle like an electron and V describes a magnetic field which acts on the particle. Then the line integral can be interpreted as computing the work done on the particle by the magnetic field as it moves.

Those are the mathematical facts but something visual can help.



Courtesy of Wikipedia: Line Integral¹

In the picture, we have a curve in yellow that is moving in some ambient space. There is a vector space which permeates through this space. As it moves from the green to red dot, we see its initial direction of movement is aligned with the direction of the vector field. But at some points, it becomes closer to being perpendicular. The green graph below the picture of the

¹https://en.wikipedia.org/wiki/Line_integral

curve and vector field shows the dot product (which intuitively, is how much the two directions are aligned). Note that at some point, the movement of the vector field is pointing opposite to the direction of the curve and hence, we have negative values. The line integral is simply the signed area under the green curve. One can imagine straightening the yellow curve out and placing it on $[a, b]$ (after some dilations) and then integrating a function like in single-variable calculus.

Example 1.1. Let $C(a)$ be a circle of radius a centered at the origin; we can parametrize it with $r(t) = \langle a \cos t, a \sin t \rangle, t \in [0, 2\pi]$. Let $V(x, y) = \langle -y, x \rangle$. We can see that it's not conservative because if it were, then we need a function ϕ so that $\phi_x = -y, \phi_y = x$. Then, taking further derivatives, $\phi_{yx} = -1, \phi_{xy} = 1$. This means that ϕ does not have continuous 2nd order partials because if it did, Clairaut's theorem says these mixed partials should equal. Now for the set up: $V(r(t)) = \langle -a \sin t, a \cos t \rangle$ and $r'(t) = \langle -a \sin t, a \cos t \rangle$. So $V(r(t)) \cdot r'(t) = a^2(\sin^2 t + \cos^2 t) = a^2$. The integral becomes easy to do: $\oint_{C(a)} V \cdot dr = \int_0^{2\pi} a^2 dt = 2\pi a^2$.

Theorem 1.2. (Fundamental Theorem of Line Integrals) Let C be a piecewise-smooth curve contained in a connected and simply connected domain on which the conservative vector field $V = \nabla\phi$ is defined. Let $r : [a, b] \rightarrow \mathbb{R}^n$ be a parametrization of C . Then

$$\int_C \nabla\phi \cdot dr = \phi(r(b)) - \phi(r(a)).$$

This means that the line integral only depends on the end points of the curve. The left hand side (LHS) is an integral which means we're, roughly speaking, summing up infinitely many infinitesimals to obtain a value. However, the RHS tells us that almost all of them cancel out except for the endpoints. Hence, if a particle moves through a conservative vector field, the net work done on the particle only depends on where it starts and ends. It could take a very direct path or take an extremely circuitous route. Moreover, if it starts and ends at the same point, then the net work is zero. We'll write this as a corollary. The symbol $\oint_C$ just emphasizes that C is a loop since it starts and ends at the same point.

Corollary 1.3. In the context of the above theorem,

$$\oint_C \nabla\phi \cdot dr = 0.$$

Proof. (Fundamental Theorem of Line Integrals): Let $r : [a, b] \rightarrow \mathbb{R}^n$ be a parametrization. The integrand is $\nabla\phi \cdot dr$ which, when written out, is

$$\sum_{i=1}^n \frac{\partial\phi}{\partial x_i} \frac{dr}{dt} = \frac{d}{dt}\phi(r(t)).$$

This equality follows from the chain rule. Then the integral becomes $\int_a^b \frac{d}{dt}\phi(r(t)) dt = \phi(r(b)) - \phi(r(a))$ by the Fundamental Theorem of Calculus. $\square$

Example 1.4. Let's tweak the vector field from our first example slightly. Recall that in polar coordinates, $\theta = \tan^{-1}(y/x)$ and so $\nabla\theta = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$. This is similar to the original vector field but note that it's now been normalized to have length 1. The flow of this vector field (like the one before), rotates the plane counterclockwise but the previous one did it more rapidly as you move away from the origin. We also note that for θ , the function is not defined on the entire y -axis where $x = 0$ but $\nabla\theta$ is only not defined at the origin. Restricting to the circle of radius a , we see that $x^2 + y^2 = a^2$ and so $\oint_{C(a)} V \cdot dr = \int_0^{2\pi} dt = 2\pi$. This may seem strange because we just integrated a conservative vector field on a closed loop; shouldn't the result be

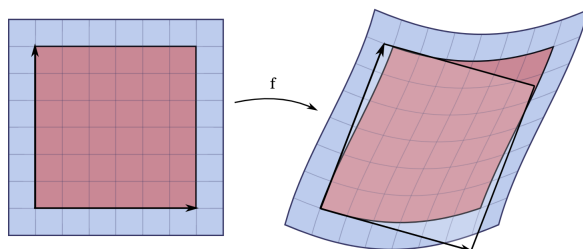
0? It must be that one of the conditions of the hypothesis is not fulfilled. Indeed, we find that $\nabla\theta$ is defined on $\mathbb{R}^2 \setminus (0,0)$ rather than $\mathbb{R}^2$; this punctured plane is not simply connected.

Remark: This simple example can be converted to the setting of meromorphic functions in complex analysis and is a simple example of the Cauchy Residue Theorem.

2 Surface Integration

Let $S \subset \mathbb{R}^3$ be a surface parametrized by $r(u, v)$; that is, we have a map $r : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ which can alternatively be viewed as a collection of three functions that are each in 2-variables. Then r is continuous, differentiable, etc if each of the functions are. An example would be to use the graph of a function $f : D \rightarrow \mathbb{R}$. Then $r(u, v) = (u, v, f(u, v))$. But we can also have something different such as $r(u, v) = (u \cos v, u \sin v, u^2)$ which is smooth and not the graph of a function.

In order to integrate a function f over a domain S , we do the usual set it up in the following way. The integral should ultimately be defined via something like $\lim_{N \rightarrow \infty} \sum^N f(x, y, z) \Delta A$. That is, we subdivide S into pieces, sample a point in each piece and evaluate f on it, and multiply each of those values with the area ΔA of the piece. Here, note that since S , even though it's a 2-dimensional space, it makes sense to evaluate f , a 3-variable function, on it because S is inside of $\mathbb{R}^3$. Usually, in the limit, $\Delta A \rightarrow dA = du dv$ but the map r may distort the domain into something with curvature and the area form will also be distorted (see the image below; in your mind, replace the f there with r).



Courtesy of Wikipedia: Jacobian²

So, how do we get an appropriate area form dA to take into account this distortion? If the map r is differentiable, then we can study the tangent spaces of the surface S at each point. Locally, the tangent space is a close approximation to the shape of S and so the coordinate u, v axes are sent to vectors r_u, r_v in the tangent space. The construction of r_u is to just take the u th partial derivative of each of the three coordinate functions; similarly for r_v . So in our example from above, $r_u = \langle \cos v, \sin v, 2u \rangle, r_v = \langle -u \sin v, u \cos v, 0 \rangle$. Anyways, these two vectors span a small parallelogram and we can find the area of the parallelogram to use it for constructing our area form dA . Since we're in $\mathbb{R}^3$, there is an easy way to compute the area via the cross product: $|r_u \times r_v|$. Hence, our $dA = |r_u \times r_v| du dv$.

This extra factor $|r_u \times r_v|$ is called the **Jacobian** and it may seem strange but we've actually seen it before in other contexts. When changing from Cartesian to polar coordinates, we have $dx dy = r dr d\theta$; the extra r has the same role. Similarly, from Cartesian to spherical, we have $dx dy dz = \rho^2 \sin \phi d\rho d\theta d\phi$ where $\rho^2 \sin \phi$ is the Jacobian. There is one caveat which I'll mention but is generally confusing for students and I won't really emphasize on exams: the surface must be orientable which intuitively means that we can distinguish two sides of the surface (this isn't always possible; the Möbius band is an example of a nonorientable surface). If it's orientable, then we can find a consistent choice of local Jacobians that patch together correctly to give a global Jacobian and hence, a global area form. For nonorientable surfaces,

²https://en.wikipedia.org/wiki/Jacobian_matrix_and_determinant

we can find global forms which are almost area forms but they vanish at certain points (and hence, fail to give a nonzero area in those locales). If a surface is orientable, there are two choices of orientation; after picking one, we say that the surface is **oriented**.

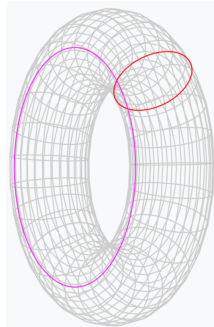
With that caveat aside, our **surface integral** of a function f over an oriented surface $S \subset \mathbb{R}^3$ parametrized by $r : D \rightarrow \mathbb{R}^3$ is denoted and defined by:

$$\int_S f dA := \iint_D f(r(u, v)) |r_u \times r_v| du dv.$$

1. If $f \equiv 1$, then we're just integrating the area form over S and we will get the surface area.
2. If a surface cannot be parametrized by a single map r , this is no issue; we can cover the surface with various maps r and then integrate over them individually and then sum them up. For example, the unit sphere can be covered by $\sqrt{1 - u^2 - v^2}$ (northern hemisphere) and by $-\sqrt{1 - u^2 - v^2}$ (southern hemisphere). Of course, if we use spherical coordinates, we can cover everything in one parametrization though the domain won't be open due to the fact that the sphere and $\mathbb{R}^2$ are not homeomorphic.

Example 2.1. Let's parametrize the torus T (or donut, shown below). There are two circular directions to go along, either the magenta or the red circle. Let R be the radius of the magenta circle and r be the radius of the red circle. For this to be an actual torus, we need $r < R$. Our parametrization is:

$$r(u, v) = ((R + r \cos u) \cos v, (R + r \cos u) \sin v, r \sin u).$$



Courtesy of Wikipedia: Torus³

Using the parametrization, we may obtain the area form. The first thing to do is produce r_u and r_v ; by differentiation, we get

$$r_u = \langle -r \sin u \cos v, -r \sin u \sin v, r \cos u \rangle, r_v = \langle -(R + r \cos u) \sin v, (R + r \cos u) \cos v, 0 \rangle.$$

After simplifying with the Pythagorean identity, their cross product is

$$r_u \times r_v = \langle -r(R + r \cos u) \cos u \cos v, -r(R + r \cos u) \cos u \sin v, -r(R + r \cos u) \sin u \rangle$$

and $|r_u \times r_v| = r(R + r \cos u)$. So the area form is $r(R + r \cos u) du dv$ and we can even get a unit normal vector field on the torus $\hat{n} = \langle -\cos u \cos v, -\cos u \sin v, -\sin u \rangle$. This is an **inward** pointing field. We may compute the surface area of T now by computing the integral

$$\int_T dA = \int_0^{2\pi} \int_0^{2\pi} r(R + r \cos u) dv du = 2\pi r \int_0^{2\pi} (R + r \cos u) du = 4\pi^2 r R.$$

³<https://en.wikipedia.org/wiki/Torus>

Note that this is the same area as a rectangle with dimensions $2\pi r \times 2\pi R$. Indeed, one way to construct a torus is to take such a rectangle and identify the opposite sides. So this parametrization of the torus, even though it has a nontrivial Jacobian to account for the curvature of the torus T , integrates in a way to suggest there is actually no curvature. This is actually a very deep fact; the curvature we see of T is an artifact of how it is embedded into $\mathbb{R}^3$; that's an extrinsic feature. However, the intrinsic curvature of T is in fact, 0 because it can be constructed from a flat rectangle. This is an example of the Theorema Egregium of Gauss.

Example 2.2. Let $g(x, y, z) = 4x^2 + 4y^2 + 1$ and S be the graph of $z = 1 - x^2 - y^2$ over the unit disk D^2 . So a parametrization of S is $r : D^2 \rightarrow \mathbb{R}^3$ using $r(u, v) = (u, v, 1 - u^2 - v^2)$. Then $r_u = \langle 1, 0, -2u \rangle$, $r_v = \langle 0, 1, -2v \rangle$, $r_u \times r_v = \langle 2u, 2v, 1 \rangle$ and so the area form is $\sqrt{4u^2 + 4v^2 + 1} du dv$. Then, $g(r(u, v)) = 4u^2 + 4v^2 + 1$ and the integral of g over S becomes:

$$\iint_{D^2} (4u^2 + 4v^2 + 1)^{3/2} dv du = \int_0^{2\pi} \int_0^1 (4r^2 + 1)^{3/2} r dr d\theta.$$

The equality here comes via polar coordinates. We may then integrate to compute the final result: $\frac{\pi}{10}(25\sqrt{5} - 1)$.

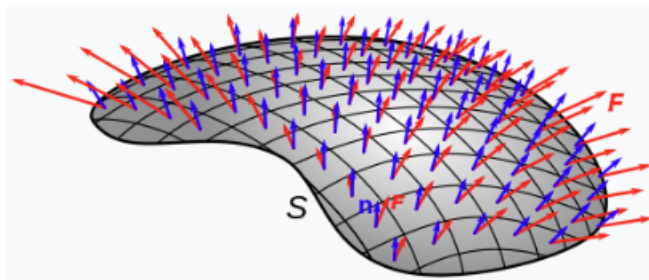
2.1 Comparing Line and Surface Integrals + Flux

Let's now compare surface and line integration. For line integrals, note that a parametrization $r(t)$ of a curve C automatically places an orientation (direction of travel) on C . In fact, $r'(t)$ gives the tangency direction and so locally, we can just use it to determine the direction of travel. This is important since if we parametrize the other way, the line integral is changed by a minus sign. For surfaces, we also need an orientation; it's secretly hiding in the fact that we have a Jacobian which is built out of derivatives.

For line integrals, when we do use a vector field V , we have a convenient way of producing a scalar-valued function: use the dot product to get $V(r(t)) \cdot r'(t)$. For surfaces, we can't use the derivatives of the parametrization $r(u, v)$ since it's a 2-variable function, not a 1-variable function. Thus, we need extra data, namely, the choice of an orientation. I've been hand-wavy about what an orientation actually is but it suffices to know that for surfaces in $\mathbb{R}^3$, an orientation is implied by having a unit normal vector field $\hat{n}$ on the surface. We then use the right-hand rule to treat $\hat{n}$ as if it were the z -axis and that tells us how the the tangent plane is oriented with x, y -axes (equivalently, polar coordinates). Being unit length everywhere means $\hat{n}$ never vanishes and being normal means the vector field is perpendicular to the surface everywhere. Again, there are two choices of such vector fields. Once $\hat{n}$ is chosen, then we may restrict a vector field V on $\mathbb{R}^3$ to S by using $V(r(u, v))$ and taking a dot product with $\hat{n}$. In summary:

$$\int_S V \cdot \hat{n} dA := \iint_D V(r(u, v)) \cdot \hat{n} |r_u \times r_v| du dv.$$

Here is a picture to show what is going on. We have a vector field F , shown in red, restricted to a surface S with a unit normal $\hat{n}$ shown in blue.



Note that the formula I gave above is very convenient but we're making use of the fact that $S \subset \mathbb{R}^3$ in two ways. First, we're using the cross product which only works (nontrivially) for $\mathbb{R}^3$ and $\mathbb{R}^7$. Secondly, we are using $\hat{n}$ for the dot product and this $\hat{n}$ induces an orientation on S only when it's inside of $\mathbb{R}^3$. It's an implication rather than equivalence because we really only need a nonvanishing transversal vector field rather than one of unit length and perpendicular to S . However, if we're interested in surfaces in other spaces, we cannot use such a formula. For example, in physics, a propagating string etches out a 2-dim surface called a worldsheet which lives in a higher dimensional space (typically, either 4, 10, 11, or 26 dimensions depending on whether we're looking at spacetime, supersymmetric string theory, M-theory, or bosonic string theory).

But at any rate, despite not being up to par for string theory, this particular kind of surface integral is actually very important for other parts of physics and is called the **flux**. This is because, in physics, we are often interested in the flux which is the measurement of what is transferred by the flow of a vector field V through a (permeable) surface S . For example, if S is some kind of metal and V describes a magnetic field, we may want to measure the total amount of transfer through the metal. If the magnetic field strength varies over time, we would then have a time-dependent function for the flux. If S is a superconductor and the Meisner effect occurs; i.e. the magnetic fields are repelled, then V vanishes around S and the flux is 0.

Example 2.3. We now consider a flux integral over the upper hemisphere of the unit sphere with orientation being a unit normal vector field pointing radially outwards. Let $V(x, y, z) = \langle x, y, z \rangle$. Let $f = \sqrt{1 - x^2 - y^2}$; it's graph is the upper hemisphere of the unit sphere so $r(u, v) = (u, v, f(u, v))$. Then $r_u = \langle 1, 0, \partial_u f \rangle, r_v = \langle 0, 1, \partial_v f \rangle$. Their cross product is then $\langle -\partial_u f, -\partial_v f, 1 \rangle$.

For our situation with f , the quotient rule gives us $\partial_u f = \frac{u}{\sqrt{1-u^2-v^2}}$ and similarly for $\partial_v f$. So then

$$r_u \times r_v = \left\langle \frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}, 1 \right\rangle.$$

The norm of this cross product is $(1 - u^2 - v^2)^{-1/2}$ so $dA = (1 - u^2 - v^2)^{-1/2} dv du$. Dividing by this norm gives a unit normal vector field $\hat{n} = \langle u, v, \sqrt{1 - u^2 - v^2} \rangle$ though since we're in $\mathbb{R}^3$, we should technically use x, y instead of u, v to describe the vector field. Let's just make that symbolic change now. We then have the information we need now to compute the integral. Let D denote the unit disk. Then:

$$\begin{aligned} \int_S V \cdot \hat{n} dA &= \int_D (\langle x, y, z \rangle \cdot \langle x, y, \sqrt{1 - x^2 - y^2} \rangle) (1 - x^2 - y^2)^{-1/2} dA \\ &= \int_D (x^2 + y^2 + z\sqrt{1 - x^2 - y^2}) (1 - x^2 - y^2)^{-1/2} dA \\ &= \int_D (x^2 + y^2 + 1 - x^2 - y^2) (1 - x^2 - y^2)^{-1/2} dA \\ &= \int_D (1 - x^2 - y^2)^{-1/2} dA \\ &= \int_0^{2\pi} \int_0^1 \frac{r}{\sqrt{1 - r^2}} dr d\theta \\ &= 2\pi \end{aligned}$$

The third equality comes from the fact that on our upper hemisphere, $z = \sqrt{1 - x^2 - y^2}$. The 5th is from switching to polar coordinates. After using substitution, we arrive at 2π as

⁴<https://en.wikipedia.org/wiki/Flux>

the final answer. Coincidentally, this happens to be the surface area of S which is the upper hemisphere.

3 Differential Operators

The **del operator** is $\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$ and, as it stands, it largely a collection of symbols. But we can pair it with other operations. For example, if $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a differentiable function, we already know that the gradient is defined by $\nabla \phi = \langle \partial_x \phi, \partial_y \phi, \partial_z \phi \rangle$. We can also pair this ∇ up with $\cdot$ and $\times$, the dot and cross products. That is, if we have a vector field $V(x, y, z) = \langle F, G, H \rangle$ on $\mathbb{R}^3$, we have the **divergence** $\nabla \cdot V = \partial_x F + \partial_y G + \partial_z H$. The **curl** is $\nabla \times V$ and is defined by:

$$\nabla \times V = \det \begin{pmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F & G & H \end{pmatrix} = \langle \partial_y H - \partial_z G, -\partial_x H + \partial_z F, \partial_x G - \partial_y F \rangle.$$

We will sometimes also use the text grad, curl, div instead of the symbols since seeing ∇ slightly modified can be confusing notation. Some remarks are now in order.

1. The divergence of a vector field can be defined in any dimensions but curl can only be defined in $\mathbb{R}^3$ due to the cross product.
2. The divergence of a vector field is a scalar function whereas the curl is another vector field.
3. The intuitive way to think of divergence is as follows. If we view a vector field as describing the flow of some physical notion like fluid or magnetic fields, then the when we plug in a point $p = (x, y, z)$ into the divergence, we get a number back which we can view as telling us the net flow of V through p . For example, if $\text{div}(V)(p) = -12$, this means that the on the whole, the vector field is flowing into the point p . If V represents fluid, then p is like a **sink**. If the divergence is positive, then we have that fluid, on the whole, is flowing out of p ; then it is viewed as a **source**. If the divergence is 0, then it is neither. If a vector field is such that the divergence is 0 at every single point, then we call it divergence free. We can imagine that it is modeling an incompressible fluid like water, compared to a compressible fluid like air. Then integrating the divergence over some solid object tells us the net flow in or out of that whole solid.
4. The curl of a vector field at a point p is a measurement of how the vector field is rotating an infinitesimally small ball centered at p . It's easier to describe this in physical terms. Suppose the vector field represents the flow of a river and we have a way of submerging a small sphere into the river so that magically, it does not shift its position. However, it can spin around based on how the forces from the water act on it. There is an overall balance of forces so that the ball rotates along some axis at some speed. The axis of rotation is exactly the vector given by the curl at the point p and the speed of the rotation is given by the length of the vector. Of course, in an actual river, the flow of water changes from moment to moment and so the curl should be time dependent. We would then see the sphere "wobble" around. Put another way, in this context, the curl measures the infinitesimal net circulation at a snapshot moment.

Example 3.1. Let $V(x, y, z) = \langle x + y, e^{xyz}, x - y + z^2 \rangle$. Then $\nabla \cdot V = 1 + xze^{xyz} + 2z$. To compute the curl takes a bit more work but $\nabla \times V = \langle -1 - xye^{xyz}, -1, yze^{xyz} - 1 \rangle$.

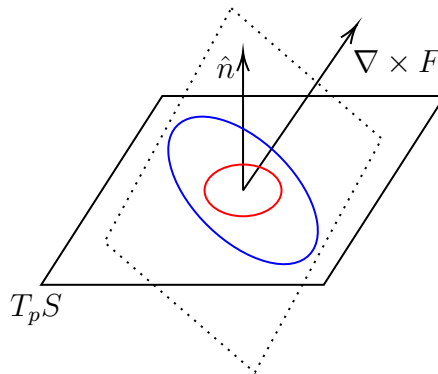
We now present two facts as a lemma which can be easily established by straightforward computation.

Lemma 3.2. *When the functions or vector fields involved are at least twice-differentiable, then $\text{curl}(\text{grad}) = 0$ and $\text{div}(\text{curl}) = 0$.*

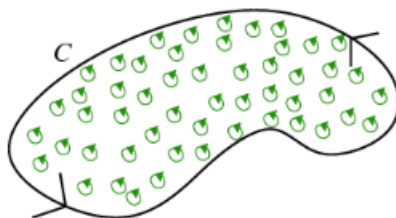
4 Green's and Stokes' Theorem

Note that locally, a surface is approximated by its tangent space, a 2-dim vector space like $\mathbb{R}^2$. Now, $\mathbb{R}^2$ has Cartesian coordinates which comes along with an orientation which amounts to having an ordered basis; we typically call the horizontal axis the x -axis and it comes before the vertical one, the y -axis. If we switch to polar coordinates, we see that the angle θ goes **counter clockwise**. This may seem unimportant to emphasize but it leads to an important point that I've already touched on. **It is always possible to find local coordinates with orientation on a surface but not always possible to make such coherent choices globally.** Famous examples of nonorientable surfaces like this include $\mathbb{RP}^2$ and the Möbius band. The former example is **closed** which means the surface does not have a boundary (or to use another term, edge) while the latter does have boundary. It turns out that all closed surfaces embedded into $\mathbb{R}^3$ must be orientable though one can immerse $\mathbb{RP}^2$ into $\mathbb{R}^3$. The Möbius band can be embedded into $\mathbb{R}^3$ but it has boundary. Anyways, the fact that having oriented basis everywhere is equivalent to having a nowhere vanishing transversal vector field should remind us of the curl. The curl gives us an axis around which there is rotation. This is very much like having a normal vector field to a tangent plane on which we have polar coordinates to describe rotation. Example 4.4 below is the paradigm of this.

So what is $\text{curl}(F) \cdot \hat{n}$ telling us when we're restricted to S ? It tells us how, infinitesimally at each point $p \in S$, the vector field is rotating points around with its flow. The plane in which the infinitesimal rotation ("microcirculation") occurs need not be the tangent plane $T_p S$ but it can still affect rotation in S . Thus, the dot product $\text{curl}(F) \cdot \hat{n}$ tells us how much the plane of rotation aligns with the tangent plane; equivalently, how much the axis of rotation aligns with $\hat{n}$. Interpreted another way, it measures how much rotating F is doing on S . In the picture below, we have the tangent plane $T_p S$ with a unit normal pointing upwards and we have another (dotted) plane perpendicular to the curl, here written with the notation $\nabla \times F$. The blue loop indicates rotation in the dotted plane and it contributes to rotation in $T_p S$. The red loop represents this rotation but it is smaller because the curl and $\hat{n}$ are not aligned very much.



With that, let's now consider what it would mean to **integrate** $\text{curl}(F) \cdot \hat{n}$ on S . In the next picture, we have a bunch of spinning loops going counterclockwise on the surface. Certainly it's possible to transition to clockwise spinning but there would be some vanishing of the curl by continuity.



Courtesy of Math Insight ⁵

If you have set two of the counterclockwise spinning loops next to each other, observe that the right side of the left loop is going up. Comparatively, the left side of the loop on the right is going down. Here's a zoomed in version of what I mean. Note that if the loops are of the same size and spinning at the same speed, if they come in contact, their spins should cancel out.

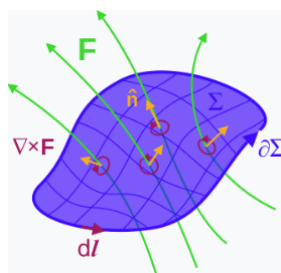


This cancellation of the spins occurs when you start adding up the curl (which is what integration is). To make this precise, we need more technical ideas but the heart of the matter has already been said. In a small region, these very small rotating loops of approximately uniform size which represent the curl of the vector field, are rotating very close to each other and also in approximately the same direction and speed. This is because the curl should vary continuously and hence, very nearby points should have very similar curl. Hence, we have cancellation on the interior of the surface. The only time we don't have cancellation is when we get to the boundary of the surface. For example, if we're on the left end of the surface on the boundary, then the loops on the surface do not have anything further to their left to cancel their rotation since moving left would take us off the surface. Moreover, since we have counterclockwise circulation with respect to the orientation, then on the boundary, that circulation pushes movement in the counterclockwise direction. So if we parametrize the boundary curve C by a counterclockwise moving $r(t)$, the orientation of $r(t)$ agrees with what is being induced onto C by the orientation of S . Moreover, as F moves near the boundary, how much it rotates depends on how much it is tangent to the boundary; this is measured by the function $F \cdot dr$ on C . This line of reasoning from the last few paragraphs is a rather hand-wavy proof of the following theorem of Stokes:

Theorem 4.1. (*Stokes' Theorem*) Let S be a piecewise smooth surface in $\mathbb{R}^3$ equipped with a choice of unit normal vector field $\hat{n}$; this induces a choice of orientation. Suppose that the boundary of S is a piecewise smooth curve C with a parametrization r that traverses C in the counterclockwise direction with respect to $\hat{n}$. If F is a vector field on an open region containing S , then

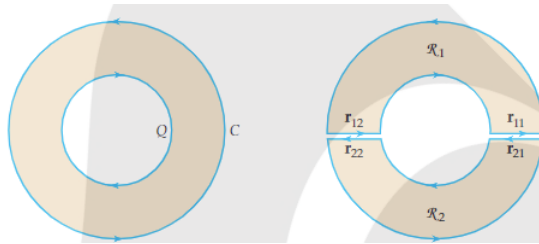
$$\iint_S (\text{curl}(F) \cdot \hat{n}) dA = \oint_C F \cdot dr$$

Here's another picture of the situation (they use Σ instead of S):



⁵https://mathinsight.org/greens_theorem_idea

One immediate thing to say is that it's very important that the orientation we place on C via the parametrization agrees with that of S . This is because the orientation of S naturally induces an orientation on C ; if we pick the wrong parametrization on C , it will be opposite to the induced orientation. The second thing to say is that we don't actually need a single boundary component; we can slice up the domain into pieces so that each piece has a connected boundary component and then use cancellation when gluing the pieces back together. Here's a picture of an annulus to show what I mean.



Courtesy of Taalman-Kohn's *Calculus*

The annulus has two boundary components: an inner and an outer circle. If we cut it in half as shown in the picture, each piece now has connected boundary. If we orient the inner circle so that it's going **clockwise** instead of counterclockwise, we see that on the two halves, the overall orientation of the boundary is again, counterclockwise. Put another way, if we were to walk on the edge of the boundary, the correct orientation is to always have the domain to our left as we walk forward. Now, note the horizontal segments denoted r_{12}, r_{22} ; they are basically the same segment, just with opposite orientations. Whatever contributions come from r_{12} , we have the exact same contribution but with opposite sign coming from r_{22} ; so they cancel. The same is true of the other pair of line segments. Hence, we can do a double integral of the curl of F to the two halves and when we add them together, we'll have the same result as if we integrated curl of F on the annulus. On the other hand, if we compute the two line integrals on the two halves, their sum will equal the two line integrals on the inner and outer circles of the annulus. This suggests that Stokes theorem is still true even for domains with multiple boundary components. Indeed, our reasoning here with the annulus applies to all other kinds of oriented surfaces in $\mathbb{R}^3$.

The third thing to say is that Stokes' theorem can be used to prove another theorem called Green's theorem. Really, it's just Stokes theorem when we study surfaces confined to $\mathbb{R}^2$ but the point is that we then have a more explicit description of the integral if we're given an explicit description of the vector field.

Theorem 4.2. (*Green's Theorem*) Let $D \subset \mathbb{R}^2$ be a region with piecewise smooth boundary being a simple closed curve C , parametrized by $r(t)$ in a counterclockwise direction. Let $F : D \rightarrow \mathbb{R}^2$ be a differentiable vector field described as $F(x, y) = \langle P(x, y), Q(x, y) \rangle$. Then

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_C \langle P, Q \rangle \cdot dr.$$

Since we're in $\mathbb{R}^2$, we can write $r(t) = \langle x(t), y(t) \rangle$ and so we sometimes see dr written as $dr = \langle dx, dy \rangle$ and hence, the RHS of Green's theorem is sometimes written as $\oint_C (P dx + Q dy)$.

Proof. By computation, $\text{curl}(F) = \langle 0, 0, Q_x - P_y \rangle$. This is because P, Q do not depend on z and hence $\partial_z P = \partial_z Q = 0$. The unit normal is $\hat{n} = \langle 0, 0, 1 \rangle$, parallel to the z -axis. Then the surface

⁶https://en.wikipedia.org/wiki/Maxwell%27s_equations

integral of $\text{curl}(F) \cdot \hat{n}$ over the region $D \subset \mathbb{R}^2$ is $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$. By Stokes' Theorem, this is equal to $\oint_C \langle P, Q \rangle \cdot dr$. $\square$

We'll now look at a few examples. The first shows that Stokes' theorem can be labor saving.

Example 4.3. Let S be the piece of the paraboloid defined by $z = 5 - x^2 - y^2$ sitting above the plane $z = 1$ with outward pointing $\hat{n}$ as orientation. Let $F = \langle z^2, -3xy, x^3y^3 \rangle$. Then, if we want to directly compute $\iint_S (\text{curl}(F) \cdot \hat{n}) dA$, this would be quite a headache. We would need to find a description of $\hat{n}$ which involves some derivatives and cross product. To compute the curl, we would need to do something similar. Then we would need to compute a double integral. However, Stokes theorem says that the result is equal to a much easier integral. Let C be the boundary of S ; it is determined by setting $1 = 5 - x^2 - y^2$ and so $x^2 + y^2 = 4$. We can parametrize C by $r(t) = \langle 2 \cos t, 2 \sin t, 1 \rangle$; the parametrization matches the induced orientation from S . Then $F(r(t)) = \langle 1, -12 \cos t \sin t, 64 \cos^3 t \sin^3 t \rangle$ and $r'(t) = \langle -2 \sin t, 2 \cos t, 0 \rangle$. The dot product of the two is $-2 \sin t - 24 \cos^2 t \sin t$. Hence, the result we want can be computed from

$$-2 \int_0^{2\pi} (\sin t + 12 \cos^2 t \sin t) dt = 0.$$

It's easy to see that the first term integrates to 0 by the periodicity of $\sin t$ and the second does as well once we use the substitution $u = \cos t$. This is because then the lower and upper u -bounds are both $1 = \cos(0) = \cos(2\pi)$.

Example 4.4. Recall our example from the line integral section where $V(x, y) = \langle -y, x \rangle$. So $P = -y, Q = x$. We'd computed that the line integral of this vector field around a circle of radius a gives a result of $2\pi a^2$. Let $D(a)$ be a disk of radius a centered at the origin; so it has C as boundary and we've oriented things so that C is parametrized by a counter clockwise direction. Let's verify Green's theorem for this situation. We already know the RHS of the equation of Green's theorem is $2\pi a^2$ so we'll compute the LHS. This is actually very easy because the integrand is $P_y - Q_x = 1 - (-1) = 2$ and so we're just integrating $\iint_D 2 dA = 2 \text{Area}(D(a)) = 2\pi a^2$. This verifies Green's theorem in this situation.

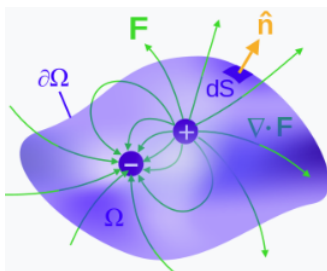
Example 4.5. Let's use the conservative vector field $\nabla \theta = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$. Then the quotient rule gives that away from the origin, $P_y = Q_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}$. Hence, if D is a region not containing the origin, the LHS of Green's theorem is just integrating 0 and hence, the total outcome is 0. On the other hand, the RHS is a line integral of a conservative vector field $\nabla \theta$ over a loop in a simply connected domain and by the Fundamental Theorem of Line Integrals, must be 0. So we see that Green's theorem is true in this case as well.

Now, in our computation, the $P_y - Q_x$ is actually the curl of the vector field and what we see is that $\text{curl}(\nabla \theta) = 0$. This is a universal fact that we stated above so this example also gives us a verification of that fact.

But if we move the region D to contain the origin, it will no longer be true! We saw that the integral $\oint_C \nabla \theta \cdot dr = 2\pi$ when C is a loop enclosing the origin. This seems weird; how could a loop that encloses the origin but could *a priori* be very far from the loop, detect anything about the issue with $1/(x^2 + y^2)$ being undefined at the origin and hence, requiring us to puncture the plane? Green's theorem sheds some light on this: the integral over the region bounded by C involves integration on all the point in the region which always have value 0 except at the origin. The origin contributes 2π . Hence, vector fields and integration can detect something about the topology of the punctured plane; it's not simply connected. If there are k punctures and a loop encloses all k of them, then the line integral will be $2\pi k$; hence, we get a count of the number of punctures as well. This might not seem so important but when studying differential equations and looking for closed orbits in a dynamical system, one can use these ideas to determine where closed orbits might occur.

5 Divergence Theorem

Above, we discussed the curl of a vector field. We'll now discuss divergence. Consider the following picture (they use Ω instead of W).



Courtesy of Wikipedia⁷

If we have a 3-dim solid $W \subset \mathbb{R}^3$, then we can integrate the divergence of a vector field over W in order to find the net flow of the vector field going in/out of W . One can imagine that if there is any flow coming out of a point, then it will be flowing into a nearby point. Hence, we have some positive outward flow that is immediately cancelled out by negative inward flow. The only place where there is no cancellation is at the boundary since there's nothing beyond it to contribute. This is essentially the statement of the Divergence Theorem.

Theorem 5.1. (*Divergence Theorem*): Suppose that $W \subset \mathbb{R}^3$ is a 3-dim region bounded by a piecewise smooth, closed, oriented surface S with outward pointing normal vector field $\hat{n}$. If F is defined on an open neighborhood of W , then

$$\iiint_W \operatorname{div}(F) dV = \oiint_S (F \cdot \hat{n}) dA.$$

The symbol $\oiint$ is meant to emphasize the fact that S itself is a closed surface; i.e. it has no boundary even though it is the boundary of W ; in symbols, we write $S = \partial W$.

Let's do some examples. The first example shows that sometimes, the triple integral is easier than the double integral.

Example 5.2. Let S be a piecewise smooth surface where the top piece of it is given by the part of the paraboloid $z = 4 - 3x^2 - 3y^2$ sitting above the unit disk; i.e. $x^2 + y^2 \leq 1$. The next piece of it is the cylinder defined by $x^2 + y^2 = 1$ with $0 \leq z \leq 1$. The bottom piece is formed by $z = 0$. So the surface looks a bit like a bullet or the nosecone of a rocket. The vector field is $F = \langle xy, -\frac{y^2}{2}, z \rangle$. We would like to compute $\oiint_S (F \cdot \hat{n}) dA$ where $\hat{n}$ is outward pointing.

But observe that since S comes in three pieces, to describe $\hat{n}$, we need to compute it on all three pieces. Then we need to do three integrals on each of the separate pieces. This sounds like a lot of work. On the other hand, $\operatorname{div}(F) = y - y + 1 = 1$ which is much easier to integrate. Lastly, the solid that S bounds is easy to describe using cylindrical coordinates. In fact, the integral over the actual cylindrical piece between $z = 0$ and $z = 1$ is just the volume of an actual cylinder: $\pi r^2 h$ where r is the radius and h the height. In our situation, $r = h = 1$ so that piece contributes π . What remains is the paraboloid solid. We can use cylindrical coordinates here to find that $z = 4 - 3r^2$. Our integral is then equal to:

$$\pi + \int_0^{2\pi} \int_0^1 \int_1^{4-3r^2} r dz dr d\theta = \pi + \int_0^{2\pi} \int_0^1 3r - 3r^3 dr d\theta = \pi + 2\pi \left(\frac{3}{2}r^2 - \frac{3}{4}r^4 \right) \Big|_0^1 = \frac{5\pi}{2}.$$

⁷https://en.wikipedia.org/wiki/Maxwell%27s_equations

Example 5.3. Let e_r be the radial vector field on $\mathbb{R}^3$ of unit length (it is a unit normal vector field to any sphere centered at the origin). Explicitly,

$$e_r(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle.$$

We'll then have $g = -\frac{GM}{|r|^2} e_r$ where $|r|^2 = x^2 + y^2 + z^2$ is the distance squared from the origin. This g can be viewed as a gravitational vector field and so G, M are constants with physical meaning. G is the universal gravitational constant and M is the mass of the gravitating object. As we get closer to the object, the distance decreases and so dividing by that distance squared increases the strength of the vector field. The math here would work equally well if we wanted to model Coloumbic forces instead. Anyways, if S is a sphere of radius R centered at the origin with outward pointing normal vector field $\hat{n} = e_r$, then

$$\oiint_S g \cdot \hat{n} dA = -GM \oiint_S \frac{1}{|r|^2} dA = -\frac{GM}{R^2} \oiint_S dA = -4\pi GM \frac{R^2}{R^2} = -4\pi GM.$$

The second equality comes from the fact that this is the radius R sphere and so the distance of any point on it to the origin is always R . Then, we can compute the integral easily by just knowing the surface area of the sphere to be $4\pi R^2$ (or use spherical coordinates). On the other hand, the radius R sphere bounds the radius R ball B and hence, the divergence theorem tells us that the integral above equals

$$\iiint_B \operatorname{div}(g) dV = -4\pi GM.$$

We might notice something odd about this. The gravitational vector field is not defined at the origin but that's not a problem when we restrict our attention to a sphere S since it doesn't contain the origin. But what is $\operatorname{div}(g)$? Since we'll integrate it over B which contains the origin, we might want to see what's going on. Let's forget about the constants; then g is just $(x^2 + y^2 + z^2)^{-3/2} \langle x, y, z \rangle$. Using the quotient rule, we have:

$$\partial_x \left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}} \right) = \frac{(x^2 + y^2 + z^2)^{3/2} - 3x^2((x^2 + y^2 + z^2)^{1/2})}{(x^2 + y^2 + z^2)^3}.$$

The partial derivatives in y, z are similar and when we add them all up to compute the divergence, we have:

$$\operatorname{div}(g) = \frac{3(x^2 + y^2 + z^2)^{3/2} - 3(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{1/2}}{(x^2 + y^2 + z^2)^3} = 0!$$

So shouldn't the integral be 0, not $-4\pi GM$? But note that this differentiation only makes sense away from the origin. So the divergence of g is 0 everywhere on B but at the origin, if we trust the formula of the Divergence Theorem (even if the hypotheses are not satisfied), it contributes $-4\pi GM$ to the final answer. Rather than give up on the statement of the theorem, we should seek a way to properly expand the Divergence Theorem to include such cases.

I'll say more on that in a moment but first, compare this situation to the situation from the line integrals section where we integrated $\nabla\theta$, a conservative vector field on a loop or alternatively by Green's Theorem, we integrated its curl on a disk containing the origin. The curl is 0 everywhere except at the origin where it's not defined. But the integration shows that it contributes 2π . In that case, we were able to detect non-simply connectedness. Here, with this situation, we're detecting the failure to be 2-connected. That is; there is a 2-sphere which does not contract to a point. This generalizes simply connectedness since loops are circles, also

known as 1-spheres. In physics terms, this idea can be used to detect theoretical monopoles. Moreover, we can push these ideas to define n -connectedness and use integration to detect the failure of say, $\mathbb{R}^{n+1} \setminus 0$ to be n -connected.

Now, back to gravity. We may write the mass $M = \iiint_B \rho \, dV$ where ρ is a function describing the density of B . So we have the following:

$$\iiint_B \operatorname{div}(g) \, dV = -4\pi G \iiint_B \rho \, dV.$$

We may be tempted to say that $\operatorname{div}(g) = -4\pi G\rho$ because they integrate to the same value. Of course, this is in general, false. For example, $\int_0^{2\pi} \cos x \, dx = \int_0^{2\pi} 0 \, dx = 0$ but $\cos x$ is not the zero function. But in our case, the divergence is 0 everywhere except at one point so the density must be concentrated and we have $\operatorname{div}(g) = -4\pi G\rho$. The proper notion of the concentration is to use the Dirac delta distribution and that's what allows us to extend the Divergence Theorem. Mathematically, this may seem strange but recall that in Newtonian physics, we do assume that objects behave like point masses. Or even in general relativity, the Schwarzschild black hole metric is a vacuum solution in the sense that there is only vacuum, no mass, away from the singularity of the black hole. All the energy (equivalently, mass), is concentrated at the singularity. So, not for the first time, physics provides motivation for certain mathematical concepts like this Dirac delta distribution to be developed.

Now, the gravitational vector field $g = -\nabla\phi$ is actually conservative; there is a gravitational potential function $\phi(x, y, z) = -4GM(x^2 + y^2 + z^2)^{-1/2}$. Then, the divergence is equal to $\operatorname{div}(-\operatorname{grad} \phi)$. Note that this isn't necessarily zero; it's only the divergence of the curl which is 0. In different symbols, we have $\nabla \cdot \nabla\phi$ which is sometimes written as $\nabla^2\phi$ or $\Delta\phi$. In either case, it is called the Laplacian of ϕ . Then, the equation $\nabla^2\phi = 4\pi G\rho$ is called the **Poisson equation**. It provides the essential summary of Newtonian gravitation in terms of a differential equation. It is entirely equivalent to Newton's inverse square law but has the advantage that it is a differential equation for a scalar quantity that may be straightforward to solve.

6 Brief Remarks on Historical Developments and Electromagnetism

Historically, the way that some of these theorems were discovered was due to physical considerations. For example, the Divergence Theorem was discovered by Lagrange in 1762 as he worked on fluid mechanics. In 1813, Gauss proved a special case of the Divergence Theorem as he studied gravitational attraction. Similarly, Stokes was both a mathematician and physicist so it's likely that the theorem which bears his name originates from physics as well.

I now want to present some ideas from electromagnetism which may motivate us towards discovering the Divergence Theorem (though this is probably not what actually happened, historically). Let's suppose we have a region in space $W \subset \mathbb{R}^3$ and a time-dependent scalar function $\rho(t, x, y, z)$ which represents the charge density. So the units would be charge/volume. Then, the total charge in W is $Q(t) = \iiint_W \rho \, dV$. For example, say that W is the planet earth. Observe that if a bunch of charged particles teleported from Calcutta to Buenos Aires, the overall charge on earth will be the same. However, classically, particles do not teleport like that. So from a physics standpoint, we'd like to impose a **conservation law** which says that the total charge can only change by having charges moving in and out of the boundary of W .

Now, a moving charge induces a current I which has units being charge/time. Let $J(t, x, y, z)$ be the current density; it's a vector field because current can flow in any direction. Since we only care about current flowing past ∂W , then J has units being current/area. Hence, we have

the following:

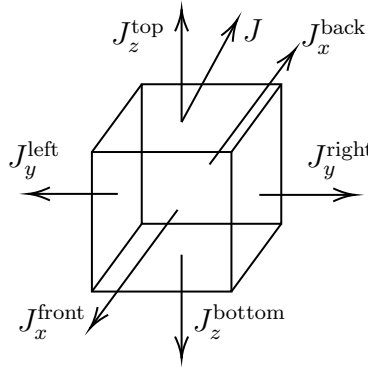
$$\frac{d}{dt}Q(t) = \iiint_W \frac{\partial \rho}{\partial t} dV = -I = -\oint_{\partial W} (J \cdot \hat{n}) dA.$$

The first equal sign comes from math; we can move $\frac{d}{dt}$ past the integration because the integration is in x, y, z only. The second equal sign is from physics: moving charges induce a current and we set this to $-I$ as we choose $\hat{n}$ to be **outward** pointing which means that charge is **leaving** W and hence, the overall charge is decreasing. The last equality is by the physical conservation law that we're imposing: total charge changes only by what moves past the boundary.

But now, we also want local conservation, not just a global one. That is, if W represents the earth, we don't want things teleporting around inside even while we say that the total charge should only change by particles moving in and out of the atmosphere. So we don't want teleportation even on short distances. This can be rephrased as saying that even for a tiny box B , we want the conservation law to hold. So let's study the situation. The conservation law would give:

$$\iiint_B \frac{\partial \rho}{\partial t} dV = - \iint_{\partial B} (J \cdot \hat{n}) dA.$$

The left integrand is roughly $\frac{\partial \rho}{\partial t} \Delta x \Delta y \Delta z$. What about the right? We have that J points out of the box and then we just want the various components of it in the x, y, z directions. Below is a picture of how current is flowing out of the box and J , the current density, captures this.



The total current out of the top face of the box is then $\Delta x \Delta y J_z^{\text{top}}$ while from the bottom, it is $\Delta x \Delta y (-J_z^{\text{bottom}})$. So adding the two together, we have $\Delta x \Delta y (J_z^{\text{top}} - J_z^{\text{bottom}})$; if we multiply by $\frac{\Delta z}{\Delta z}$, then we get $\Delta x \Delta y \Delta z (\frac{J_z^{\text{top}} - J_z^{\text{bottom}}}{\Delta z}) = \Delta x \Delta y \Delta z \frac{\partial J_z}{\partial z}$. Similarly when considering the left and right faces as well as front and back faces, we'll have similar expressions. Summing up, we'll have $\Delta x \Delta y \Delta z (\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}) = \Delta x \Delta y \Delta z \text{div}(J)$.

Hence, dividing out by the volume $\Delta V = \Delta x \Delta y \Delta z$, we have the following local equation which we'll call the local conservation law:

$$\frac{\partial \rho}{\partial t} = -\text{div}(J).$$

Note then that this leads us to a discovery of the Divergence Theorem because if we just make a substitution, we have. Any region W is built out of tiny boxes and so requiring that the conservation law hold locally would then amount to the formula of the Divergence Theorem being true for **any** region W , no matter how big or small. This equation is also called the **Continuity Equation** since it forbids teleportation. If we interpret it physically, it's saying that if the charge density decreases overtime, then the divergence must be positive and hence, the current density must be spreading out. If the charge density is increasing, then the divergence must be negative and the current density must be such that current is collecting.

Let's now look at some of the theorems from electromagnetism. On the left, we have laws in integral form and on the right, we have them in terms of differential equations. Since the integrals should hold on infinitesimally small regions because of our various theorems like Stokes' and Divergence theorem, then the local statements of the differential equations are equivalent to the integral formulation.

Gauss' Law

$$\oiint_{\partial\Omega} (E \cdot \hat{n}) dA = \frac{1}{\epsilon_0} \iiint_{\Omega} \rho dV \quad \nabla \cdot E = \frac{\rho}{\epsilon_0}$$

Gauss' Law for Magnetism

$$\oiint_{\partial\Omega} (B \cdot \hat{n}) dA = 0 \quad \nabla \cdot B = 0$$

Maxwell-Faraday Equation for Induction

$$\oint_{\partial S} E \cdot dr = -\frac{d}{dt} \iint_S (B \cdot \hat{n}) dA \quad \nabla \times E = -\frac{\partial B}{\partial t}$$

Ampere's Law

$$\oint_{\partial S} B \cdot dr = \mu_0 \left(\iint_S (J \cdot \hat{n}) dA + \epsilon_0 \frac{d}{dt} \iint_S (E \cdot \hat{n}) dA \right) \quad \nabla \times B = \mu_0 \left(J + \epsilon_0 \frac{\partial E}{\partial t} \right)$$

7 Unifying Theory

Let's now list all the important theorems that we've learned, starting from the Fundamental Theorem of Calculus. We'll use slightly different notation, namely use ∂ to mean the boundary and present the theorems in a slightly unorthodox way.

Fundamental Theorem of Calculus

$$\int_a^b f'(t) dt = \int_{\partial[a,b]} f d\mu := f(b) - f(a)$$

Fundamental Theorem of Line Integrals

$$\int_C \nabla \phi \cdot dr = \int_{\partial[a,b]} \phi(r(t)) d\mu := \phi(r(b)) - \phi(r(a))$$

Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \oint_{\partial D} \langle P, Q \rangle \cdot dr$$

Stokes' Theorem

$$\iint_S (\text{curl}(F) \cdot \hat{n}) dA = \oint_{\partial S} F \cdot dr$$

Divergence Theorem

$$\iiint_W \text{div}(F) dV = \oiint_{\partial W} (F \cdot \hat{n}) dA$$

First, note that if $F = \nabla \phi$, then Stokes' theorem plus the fact that $\text{curl}(\nabla \phi) = 0$ tells us that $\oint_{\partial S} \nabla \phi \cdot dr = 0$; this is a corollary of the Fundamental Theorem of Line Integrals. Secondly, if G is any arbitrary vector field and $F = \text{curl}(G)$, then the Divergence Theorem plus the fact that $\text{div}(\text{curl}) = 0$ tells us that $\oiint_{\partial W} (\text{curl}(G) \cdot \hat{n}) dA = 0$. Combine this with Stokes' theorem

and we have that $\oint_{\partial^2 W} G \cdot dr = 0$. But G could be any arbitrary vector field including one which is dr on the curve $C = \partial^2 W$ and then the integral $\oint_C G \cdot dr = \oint_C |r'(t)|^2 dt$ should be positive. This leads us to conclude that $C = \partial^2 W = \emptyset$ is in fact, empty.

If we interpret a 0-dim integral as simply a finite sum using the counting measure $d\mu$, then $f(b) + (-f(a))$ is treated as a 0-dim integral. So then, in all these theorems, the LHS involves an integral of an n -variable function over an n -dim domain while the RHS involves an $(n - 1)$ -dim integral. The LHS involves functions which are constructed from derivatives of functions, integrated over some domain while the RHS involves the functions themselves (no derivatives), restricted to the boundary of the domain. So there are some common themes for these theorems. One might wonder if they're all just special cases of a single theorem. The answer is yes; there is a theorem also attributed to Stokes of this nature and in fact, it works not just for $n = 1, 2, 3$ but for all dimensions n . To make sense of it, we need some more technology, however. The sequel is beyond the scope of the class.

Let M be a smooth manifold, possibly with boundary. Let $TM \rightarrow M$ denote its tangent bundle and $T^*M \rightarrow M$ its cotangent bundle. The k th exterior bundle $\Lambda^k TM$ is the bundle where each vector space over a point $p \in M$ is the k th exterior vector space of TM ; similarly for the cotangent bundle. As a reminder, if V is a vector space, then we can form tensor products $V^{\otimes k}$; the k th exterior algebra takes this tensor product and imposes skew-symmetric relations. A differential k -form on M is a section of the bundle $\Lambda^k T^*M \rightarrow M$.

If $\dim M = n$, then a volume form is a nowhere vanishing differential n -form. Its existence implies orientability of M and an orientation on M is a conformal class of volume form; there are two choices when M is orientable. On manifolds, specifically on differential forms, there is an important operator called the exterior derivative d . It can be applied to all differential forms, regardless of what k is and it has the feature that $d^2 = 0$. Examples of d that we've seen are the gradient, curl, and divergence and the fact that $d^2 = 0$ implies that $\text{curl}(\text{grad}) = 0$ and $\text{div}(\text{curl}) = 0$. Also, we can integrate k -forms on k -dim manifolds.

Now, differential forms give a cochain complex $(\Omega^*(M), d)$ and the cohomology is called de Rham cohomology $H_{dR}^*(M)$ which is Poincaré dual to singular homology tensor $\mathbb{R}$. The singular chain complex of M with $\mathbb{R}$ coefficients is $(C_*(M), \partial)$. It is no accident that just as $d^2 = 0$, we have that $\partial^2 = 0$; i.e. the boundary of the boundary of something is empty. One of the facts is that if M is a closed n -manifold (no boundary), then an exact volume form integrates to 0 and in general, we have an isomorphism: $\int_M : H_{dR}^n(M) \rightarrow \mathbb{R}$. If instead, M is oriented with boundary ∂M , then we have the following theorem called the Stokes theorem.

Theorem 7.1. *Let M be an oriented n -manifold with boundary ∂M that has the induced orientation. Let ω be an $(n - 1)$ -form on M . Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

This concise formula recovers the theorems we see above on surfaces and volumes in $\mathbb{R}^3$. Those theorems rely on a metric (the dot product) in order to not have to use the language of differential forms. The metric gives an isomorphism between the tangent and cotangent bundle and hence, we may identify 1-forms with vector fields. Moreover, the metric also gives us something called the Hodge- $*$ operator and hence, we can obtain a 1-form from the Hodge- $*$ of a 2-form then apply the isomorphism to get a vector field. This is what curl does. Lastly, if we have a 2-form, we apply these two operations to get a vector field. Then, the divergence gives us a function which, when we apply Hodge- $*$, gives a 3-form that we can integrate. Above, I'd mentioned the idea of a divergence free vector field. If we're on an n -manifold with volume form ω , then the divergence of V is defined as follows. We have that $d\iota_V \omega$ is equal to a function times ω . We call that function the divergence of V . So then, V is divergence free if $d\iota_V \omega = 0$.

This implies that the Lie derivative $\mathcal{L}_V \omega = d\iota_V \omega + \iota_V d\omega = 0$ since the second term is always zero for volume forms. Hence, V does not expand nor contract the volume form and can be viewed as representing some kind of incompressible flow.

I haven't defined the flow of a vector field exactly but I'll do that now. Let V be a vector field on a manifold M . Then, we want a family of diffeomorphisms $\phi_t : M \rightarrow M$ satisfying $\frac{d}{dt}\phi_t = V \circ \phi_t$. Due to the existence and uniqueness of solutions to ODEs, there exists such a family which we call the **flow** of V . On a closed manifold M , the flow exists for all time t and satisfies a group law: $\phi_{s+t} = \phi_s \circ \phi_t$ for $s, t \in \mathbb{R}$.

Some people call the theorem I wrote above the generalized Stokes theorem but to me, there is one more generalization of Stokes as follows. Consider a smooth submersion $\pi : X \rightarrow B$ of smooth manifolds where X has boundary. Suppose $\pi|_{\partial X}$ is also a submersion. Then this means we have a fiber bundle and we may integrate out the fiber; i.e. have the pushforward $\pi_!$. Let α be any differential form on X . We have the following result: $d(\pi_! \alpha) = \pm \pi_!(d\alpha) \mp (\pi|_{\partial X})_!(\alpha)$. When B is a point, we recover the usual Stokes theorem because the pushforward of $d\alpha$ onto a point is just zero.