Notes on Symplectic (Co)homology and Wrapped Lagrangian Floer Homology

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1 Basic Contact Geometry

Let Y^{2n+1} be an odd dimensional manifold equipped with a 1-form θ which has the property that $\theta \wedge (d\theta)^n \neq 0$ is a nonvanishing top form. By Fröbenius' theorem, this means that the distribution $\xi := \ker \theta$ is maximally non-integrable.

This condition above is equivalent to requiring that $d\theta$ is a nondegenerate 2-form on ξ. Note the similarities to $d\theta$ being symplectic.

Now, we're able to define a vector field X_R called the **Reeb vector field** in a unique way on Y: X_R satisfies $\iota_{X_R} d\theta = 0$ and $\theta(X_R) \equiv 1$. In some sense, the first condition says that X is "symplectically" orthogonal to ξ; this follows from $d\theta$ being nondegenerate on ξ. X_R is unique because of the second condition which normalizes its length.

Next, though ξ is maximally nonintegrable, we can find lower dimensional submanifolds with tangent bundle contained in ξ. In particular, a **Legendrian submanifold** $L \subset Y$ is a submanifold that satisfies $TL \subset \xi$ and has dimension n.

Claim: Legendrian submanifolds are integral submanifolds of ξ of maximal dimension.

Proof. Recall that $d\theta(X, Y) = X\theta(Y) - Y\theta(X) - \theta([X, Y])$; this is true for any 1-form θ. For any distribution D to be integrable, we can equivalently check if it is involutive (using Fröbenius' Theorem): if $X, Y \in D$, then $[X, Y] \in D$ as well.

In this case, we're looking at $\ker \theta$. So $[X, Y] \in \ker \theta$ if and only if $d\theta(X, Y)$ for all $X, Y \in \ker \theta$. The maximal dimension therefore, must be n because at each point, we're looking for a Lagrangian subspace in the symplectic vector space $\ker \theta$ with symplectic form $d\theta$. $\square$

Note that the Reeb vector field is transverse to any Legendrian submanifold L in the sense that it is not tangent to L. Also, $\mathcal{L}_{X_R}\theta = d\iota_{X_R}\theta + \iota_{X_R} d\theta = 0$. That is, the Reeb vector field preserves θ. In more specific terms, if φ_t is the flow if X_R, $\varphi_t^*\theta = \theta$. Then, if we transport a Legendrian submanifold around by the Reeb flow, it remains a Legendrian submanifold.

This, I believe, lends to the ideas behind Legendrian contact homology: given a Legendrian knot K in some contact 3-manifold, one looks for integral curves of the Reeb vector field with both endpoints on K. These are called **Reeb chords**.

The **symplectization** of Y is the symplectic manifold $(M = Y \times \mathbb{R}_t, \omega = d(e^t\theta))$. it is not too hard to check that $d(e^t\theta) = e^t(dt \wedge \theta + d\theta)$ is a symplectic form. Moreover, consider $L \times \mathbb{R}$ where $L \subset Y$ is Legendrian. In light of the claim above, one may check that $L \times \mathbb{R}$ is a Lagrangian submanifold of (M, ω). Moreover, if we sweep L along by X_R, this gives a Lagrangian submanifold as well: $\{\varphi^t(L)\}_t$. I think these are the only Lagrangians in $Y \times \mathbb{R}$; the only way to have a mix is for X_R to point along the ∂_t direction but that can't happen.

2 Boothby-Wang Fibrations

We first begin with a definition.

Definition 2.1. *Let (Y, θ) be a contact manifold. We say that θ is a **regular contact structure** if at each point $y \in Y$, there exists a local coordinate chart $(y^1, ..., y^{2n+1})$ such that every integral curve of the Reeb vector field which passes through the neighborhood does so only once.*

Note that this implies that the integral curves are all closed sets. Let us now assume that Y is compact; so the integral curves must be compact as well and thus, are homeomorphic to S^1. We then have a global S^1 action on Y since the Reeb vector field never vanishes. Moreover, in this setting, it was shown by Palais that the space B of orbits is a smooth manifold.

2.1 Main Theorem of Boothby-Wang

Therefore, we have a theorem by Boothby-Wang:

Theorem 2.2. *Let (Y, θ) be a compact, regular, contact manifold and B be the space of orbits. Then Y is a principal circle bundle over B and θ is a connection 1-form in this bundle. The curvature form ω of θ defines a symplectic form on B and determines an integral cocycle.*

Conversely, let B be a compact $2n$-dimensional symplectic manifold whose symplectic form ω determines an integral cohomology class. Then there exists a principal circle bundle $\pi : Y \to B$ and a connection form θ in it which is a contact form on Y. Moreover, the Reeb vector field of θ generates the right translations of the structural group S^1 of this bundle.

An obvious example is that of the Hopf fibrations $S^{2n+1} \to \mathbb{C}P^n$. Let's give a sketch of the proof.

Proof. Let $\pi : Y \to B$ be the map which sends a point in Y to the orbit that it's in under the Reeb flow φ_t. Since θ is a regular contact structure, we have around any point $x \in Y$, coordinates $(x^1, ..., x^{2n+1})$ on a neighborhood U_i which allow us to think of an integral curve as having $x^1, ..., x^{2n}$ fixed at some constants and just have x^{2n+1} vary.

Y is compact so we may take a finite number of U_i; then $\pi(U_i)$ serve as an open cover of B. Let's describe the bundle structure. We may do so by finding local trivializations which we construct from sections $s_i(x^1, ..., x^{2n}) = (x^1, ..., x^{2n}, c)$ where c is some constant. Then let $\Phi_i : \pi(U_i) \times S^1 \to Y$ be defined by $(x, t) \mapsto \varphi_t(s_i(x))$.

Since the Lie algebra of S^1 is $\mathbb{R}$, a connection 1-form is found in $\Omega^1(Y, \mathbb{R}) \otimes \mathbb{R}$. Let ∂_s be a basis for $\mathbb{R}$; we will still write θ for $\theta \otimes \partial_s$. To show that θ is a connection 1-form, we need to show that θ satisfies:

1. For each $p \in Y$, $g \in S^1$, θ transforms via the adjoint representation when we act on it by right multiplication: $\theta_{pg}(v \cdot g) = g^{-1} \cdot \theta_p(v) \cdot g$.

   However, S^1 is abelian and so the adjoint representation is trivial. Also, action by $t := g$ is $v \cdot t = d\varphi_t(v)$. So $\theta_{pt}(v \cdot t) = \varphi_t^* \theta_p(v) = \theta_p(v)$.

2. For each $p \in Y$, $R_p : S^1 \to Y$ is the map $g \mapsto p \cdot g$. Then, $R_p^* \theta = \omega_{MC} \in \Omega^1(S^1, \mathrm{Lie}(S^1))$, the Mauer-Cartan form which is the unique form that is invariant under left multiplication and is the identity linear map on $T_e S^1 \to \mathrm{Lie}(S^1)$.

   Since θ is invariant under the Reeb flow, it is certainly invariant under left multiplication. Moreover, since $\theta(X_R) \equiv 1$, then this normalization gives us an identity linear map on $T_e S^1 \to \mathrm{Lie}(S^1)$.

Note: I wrote things with more general notation to show that these are the criteria to check in general for connection 1-forms on principal G-bundles.

Next, let $\tilde{\omega}$ be the curvature form of θ. This is normally $\tilde{\omega} = d\theta + [\theta \wedge \theta]$. But the Lie bracket of $\mathbb{R}$ is trivial so we have $\tilde{\omega} = d\theta$. Moreover, $d\theta$ is horizontal and invariant under the action of S^1 so there exists ω such that $\tilde{\omega} = \pi^*\omega$. We have that $\pi^* d\omega = d\pi^*\omega = d\tilde{\omega} = d^2\theta = 0$. Since π is projection and $\tilde{\omega}$ is horizontal, this means that $d\omega = 0$. Moreover, $\pi^*\omega^n = (d\theta)^n \neq 0$ which is nonvanishing. Thus, ω^n is non-vanishing and therefore a symplectic form. A result of Kobayashi shows that $\omega \in H^2(B, \mathbb{Z})$.

Conversely, principal S^1-bundles over B are classified by $H^2(B, \mathbb{Z})$. This is because $S^1 = U(1)$ and the classifying space $BU(1) = \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, an Eilenberg-MacLane space. Then, homotopy classes of maps $[B, K(\mathbb{Z}, 2)] = H^2(B, \mathbb{Z})$.

Let $[\omega] \in H^2(B, \mathbb{Z})$ be our symplectic form and $P \to B$ be the corresponding principal S^1-bundle. Then let θ' be a connection 1-form on P and ω' be a 2-form such that $d\theta' = \pi^*\omega'$. But $[\omega'] = [\omega]$ so $\omega = \omega' + d\eta$. Let $\theta = \theta' + \pi^*\eta$. One can show that θ is invariant under right translations and also $d\theta = \pi^*\omega$.

Moreover, if X is a vertical vector field and $\theta'(X) \equiv 1$, the same holds for $\theta(X) \equiv 1$. Lastly, to show that θ is a contact form, pick any point and a basis for the horizontal vectors, say $(v_1, ..., v_{2n})$. Then we can show that $\theta \wedge (d\theta)^n_x(v_1, ..., v_{2n}, X) \neq 0$. $\square$

2.2 "Why do we need an integral class?"

I've often asked, "Why do we need an integral classes?" In Donaldson's theorem from his paper *Symplectic Submanifolds and Almost Complex Geometry*, he requires that $[\omega] \in H^2(M, \mathbb{Z})$. At least here, there is one interpretation.

Consider a disk $D \in B$ and let γ be its boundary loop. If we horizontally lift γ to a path $\tilde{\gamma}$ in Y, we find that the lift is unique once we set an initial condition: $\tilde{\gamma}(0) = y_0$ and $\pi(y_0) = \gamma(0)$. We know that $\tilde{\gamma}(1)$ is in the same fiber as its starting point but it may not be the same point. This is measured by the holonomy of our connection θ. Recall that holonomy can be computed by integrating a $\mathfrak{g}$-valued connection 1-form along the path and then using the exponential map to get a value in G. Another view in our setting is that this lifted path picks up some action in the Reeb vector field direction; it's been off set from its starting point.

However, let's suppose that γ is contained in a neighborhood U with which we can locally trivialize the bundle. As such, we obtain a local section σ. Then

$$\mathrm{hol}_\gamma(\theta) = \exp(2\pi i \int_{\tilde{\gamma}} \theta) = \exp(2\pi i \int_\gamma \sigma^*\theta).$$

Since this is a section, note that $(\pi \circ \sigma)^* = \sigma^* \circ \pi^* = \mathrm{id}$. So then $\pi^*\omega = d\theta$ implies that $\omega = (\pi \circ \sigma)^*\omega = \sigma^* d\theta = d\sigma^*\theta$. In other words, the integral above is equal, by Stoke's theorem, to

$$\int_\gamma \sigma^*\theta = \int_D \omega,$$

the symplectic area of the disk. This means that the symplectic area tells us how much Reeb action we pick up. Of course, one wonders about different capping disks (which are 2-chains but not 2-cycles). Suppose we have $\hat{D}$ as another capping disk for γ. If we glue them along γ, we get a 2-cycle; in this case, a 2-sphere. If ω is integral, then ω integrated on the sphere will be an integer k. This means that the holonomy does not depend on the capping disk:

$$\mathrm{hol}_\gamma(\theta) = \exp(2\pi i \int_{\hat{D}} \omega) = \exp(2\pi i(k + \int_D \omega)) = \exp(2\pi i \int_D \omega).$$

If γ is not in a neighborhood over which we can locally trivialize, we can still patch together multiple charts U_i using different sections σ_i and this result should hold.

2.3 Contact tori do not admit Boothby-Wang fibrations

Claim: Odd dimensional tori do not admit regular contact structures and therefore, are not the total space of some Boothby-Wang fibrations.

Proof. Let's suppose that we do have a fibration

$$\begin{array}{ccc} S^1 & \longrightarrow & T \\ & & \big\downarrow{\scriptstyle \pi} \\ & & B \end{array}$$

where T is a $(2n+1)$-torus. If B is also a torus, then this is a trivial fibration and the contact form α on T would give an integrable distribution $\ker\alpha$ with $B = T^{2n}$ as the integrable submanifold. But this is a contradiction as $\ker\alpha$ should be maximally non-integrable. Another way of putting this is that the curvature would have to be $d\alpha = \omega = 0$ since curvature measure integrability of the horizontal distribution.

Suppose now that we do not know anything about B. But because of the long exact sequence of homotopy groups for fibrations, we know that $\pi_k(T) = \pi_k(B) = 0$ for $k \geq 2$. And we also have

$$0 \to \pi_2(B) \to \mathbb{Z} \to \mathbb{Z}^{2n+1} \to \pi_1(B) \to 0.$$

This means that $\pi_2(B)$ is a subgroup of $\mathbb{Z}$ and so isomorphic to 0 or $\mathbb{Z}$. If it is isomorphic to $\mathbb{Z}$, then B has $K(\mathbb{Z}, 2)$ as a universal cover. But this can't be as B is finite dimensional and $K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$ is not.

So $\pi_2(B) = 0$ which implies that $\mathbb{Z}^{2n+1} \cong \mathbb{Z} \rtimes \pi_1(B)$ which shows that $\pi_1(B) = \mathbb{Z}^{2n} \oplus C$ where C is some finite cyclic group. We can in fact show that $C = 0$ (see below for lemma; it shows that if $C \neq 0$, B must have infinite rank homology). In this case, $B \simeq K(\mathbb{Z}^{2n}, 1)$, a homotopy torus.

On the other hand, the Gysin sequence tells us that we have a long exact sequence on cohomology:

$$... \to H^*(B) \to H^*(T^{2n+1}) \to H^{*-1}(B) \to H^{*+2}(B)...$$

with the connecting map (the last arrow) being wedging by the Euler class, a 2-form. In this case, the bundle is nontrivial because the curvature (the symplectic form) is nonzero. so the Euler class which is the 1st Chern class must be nonzero. So the map $H^0(B) \to H^2(B)$ is injective. In particular, we have this sequence in which the last arrow has $\ker = 0$:

$$0 \to H^1(B) \to H^1(T^{2n+1}) \to H^0(B) \to H^2(B)...$$

So then, we have that $H^1(B) \cong \mathbb{Z}^{2n+1}$. So $H^1(B)$ has rank $2n$ and $2n+1$. A contradiction. □

Lemma 2.3. *Let G be a nontrivial finite group. Then there are infinitely many i such that $H_i(BG, \mathbb{Z}) \neq 0$. In particular, BG is infinite dimensional*

Proof. By the long exact sequence on fibrations, we find that $\pi_{k+1}(BG) \cong \pi_k(G)$ for all k but more over, $\pi_k(G) = 0$ for $k \geq 1$. So $BG \simeq K(G, 1)$ in this case.

Now, EG, the universal bundle is contractible and so its Euler characteristic is $\chi(EG) = 1$. Now, suppose $H_*(BG)$ is finite. Then its Euler characteristic is an integer: $\chi(BG) \in \mathbb{Z}$. But also, by the multiplicativity of χ, $\chi(BG) \cdot \chi(G) = \chi(EG) = 1$. $\chi(G) = |G|$ and so $\chi(BG) = 1/|G|$ which is not an integer. This is a contradiction. So $H_*(BG)$ must be infinite. □

2.4 Sasakian Manifolds

Now, it is always possible to find a Riemannian metric g on Y such that $g(X_R, \cdot) = \theta$. We call (Y, θ, g) a **metric contact manifold**. Moreover, suppose that X_R is a Killing vector field. Letting ∇ be the Levi-Civita connection of g, we define a smooth section of $\mathrm{End}(TY)$ of rank $2n$ by $\varphi = \nabla_{X_R}$.

One can show that $\varphi^2 = -\mathrm{id} + \theta \otimes X_R$, $\varphi(X_R) = 0$, and $g(\varphi(X), \varphi(Z)) = g(X, Z) - \theta(X)\theta(Z)$. Moreover, if $(\nabla_X \varphi)Z = \theta(Z)X - g(X, Z)X_R$ is satisfied on Y, then (Y, θ, g) is called a **Sasakian manifold**.

On the product $Y \times \mathbb{R}$, define an almost complex structure by $J(Z, \partial_s) = (\varphi(Z) - \theta(Z)X_R, \theta(Z)\partial_s)$. Then the product metric is Kähler with respect to this J if the structure on Y is Sasakian.

The standard contact structure on S^{2n+1} is Sasakian. Since the odd betti numbers of Kähler manifolds are even, Sasakian manifolds must have their 1st betti number be even. With some more work, one can show:

Theorem 2.4. *The total space of the Boothby-Wang fibration is Sasakian if and only if the base space is a Hodge manifold.*

3 Liouville Domains

Most of this section and the next is based on A. Oancea's survey paper on symplectic homology.

3.1 Overview

Definition 3.1. *A (compact) hypersurface Σ in a symplectic manifold (M, ω) is said to be of **contact type** if there is a vector field X defined in a neighborhood of Σ, transverse to Σ and satisfies $\mathcal{L}_X \omega = \omega$. X is called the **Liouville vector field** and $\lambda = \iota_X \omega$ is the **Liouville form**. If X is globally defined, then we say Σ is of **restricted contact type***

The boundary of a compact symplectic manifold M is said to be of (restricted) contact type if the above conditions are satisfied and X is outward pointing.

Suppose we're in the situation with $\Sigma = \partial M$. Note that since ω is closed, Cartan's formula tells us that $\mathcal{L}_X \omega = d\iota_X \omega = d\lambda = \omega$. So ω is exact with λ as a primitive. Moreover, $\mathcal{L}_X \lambda = d\omega(X, X) + \iota_X \omega = \lambda$.

If we restrict λ to Σ, we find that it is a contact form since $\lambda \wedge \omega^{n-1}$ is nonvanishing on Σ. In general, if (Y, α) is a contact manifold with $\xi = \ker \alpha$ being the contact structure, then consider a diffeomorphism $\psi : M \to M$. ψ is a contactomorphism if $d\psi(\xi_x) = \xi_{\psi(x)}$.

Claim: ψ is a contactomorphism if and only if $\psi^* \alpha = e^h \alpha$ where $h : Y \to \mathbb{R}$ is some function.

Proof. To see this, suppose $v \in \ker \alpha_x$. Then of course, $\psi^* \alpha_x(v) = 0$. This leaves only a 1-dim subspace of $T_x Y$ on which $\psi^* \alpha$ is nonvanishing. Since α is only nonvanishing on this subspace, we see that $\psi^* \alpha_x = C_x \alpha$; some nonzero constant C_x. Then as x varies, we see C_x can vary however you like so long as it is always nonzero. Thus, $\psi^* \alpha = e^h \alpha$ for some function $h : Y \to \mathbb{R}$. □

Let's go back to our symplectic setting and make a few other remarks about the Liouville field X. If ψ_t is the flow of X, then we have that

$$\frac{d}{dt} \psi_t^* \omega = \psi_t^* \mathcal{L}_X \omega = \psi_t^* \omega.$$

Thus, $\psi_t^*\omega = e^t\omega$. This means that the Liouville flow exponentially expands volume. We will also later see in detail why we ask that X be outward pointing in the case that $\Sigma = \partial M$. But the short answer is that this gives us a holomorphic pseudo-convexity. The Floer trajectories will not be allowed to have a tangency point on Σ (this keeps them bounded away from Σ).

3.2 Back to Reeb Vector Field

Let's consider closed orbits of arbitrary period on a contact type level Σ. If we restrict ω to Σ, it has a 1-dim kernel for dimension reasons on which λ doesn't vanish. To see this, just consider $\mathbb{R}^4$ with symplectic form $\omega = dw \wedge dx + dy \wedge dz$ and take the hypersurface $\Sigma = \mathbb{R}^3$ with coordinates (w, y, z). Then note that restricted to Σ, $\omega(\partial_w, \cdot) = 0$ because we have no x directions; so $\partial_w \in \ker \omega|_{T\Sigma}$. However, the Liouville vector field X is transverse to Σ and can be thought of being in the x direction in a neighborhood of Σ. So then $\lambda = \iota_X\omega$ is proportional to dw and thus, does not vanish on $\ker \omega|_{T\Sigma} = \langle \partial_w \rangle$.

Now, if Σ is a regular level of an autonomous Hamiltonian H, then $X_H \in \ker \omega|_{T\Sigma}$ and $\lambda(X_H) \neq 0$. Just like before, the Reeb vector field X_R of Σ satisfies two properties: $X_R \in \ker \omega|_{T\Sigma}$ and $\lambda(X_R) \equiv 1$. An orbit of X_R is often called a **characteristic**.

Let φ_t be the flow of the Liouville field X. Then a neighborhood $\mathcal{V}$ of Σ is foliated by hypersurfaces $\{\varphi_t(\Sigma)\}_{t\in(-\delta,\delta)}$ for $\delta > 0$ small enough. In view of $\varphi_t^*\omega = e^t\omega$, we can make a coordinate change via a symplectomorphism:

$$\Psi : \Sigma \times [1-\delta, 1+\delta] \to \mathcal{V}; \Psi(p, S) = \varphi_{\log(S)}(p).$$

Observe that $\Psi^*\lambda = S \cdot \lambda|$ where $\lambda|$ means λ restricted to Σ. The reason for doing this is to realize X as $\partial/\partial S$ in a neighborhood of Σ. Then, observe that if $H : \Sigma \times [1-\delta, 1+\delta] \to \mathbb{R}$ is our Hamiltonian which only depends on S; i.e. $H(p, S) = h(S)$, then we can relate X_H with X_R: $dH(\partial_S) = dh(\partial_S) = \omega(X_H, \partial_S)$; since $X = \partial_S$ in this neighborhood, then $h'(S) = dh(\partial_S) = \omega(X_H, \partial_S) = -\lambda(X_H)$. So $\lambda(X_H) = -h'(S)$ while $\lambda(X_R) \equiv 1$. Since both X_H and X_R are in the 1-dim kernel of $\omega|_{T\Sigma}$, X_H and X_R are linearly dependent and in fact, $X_H = -h'(S)X_R$.

In general, even if H is not of this special type, we can still say that the (closed) orbits of the flows of X_H and X_R are in 1-1 correspondence since the two vector fields are linearly dependent. This is remarkable because so long as H realizes Σ as a regular level, the Hamiltonian dynamics only depend on Σ and not on H. Of course, the characteristics foliate Σ; that's just a consequence of the existence and uniqueness of integral curves from basic ODE.

3.3 Stein and Weinstein Domains

This discussion follows McDuff-Salamon's *Introduction to Symplectic Topology*.

A **generalized Morse function** $f : W \to \mathbb{R}$ is a smooth function in which all its critical points are either nondegenerate or **embryonic**; i.e. $p \in \operatorname{Crit} f$ is embryonic if there are coordinates where $f = p + x_1^3 - \sum_{i=2}^k x_i^2 + \sum_{i=k+1}^n x_i^2$. Sometimes, these are called **birth-death singularities**.

Definition 3.2. *A **Weinstein manifold** is a quadruple (W, ω, X, f), consisting of a non-compact connected symplectic manifold (W, ω) without boundary, a generalized Morse function $f : W \to \mathbb{R}$ that is bounded below and proper, and a complete Liouville vector field X on W that satisfies:*

1. *The inequality $df(X) > 0$ holds on $W \setminus \operatorname{Crit} f$.*

2. *Choose a Riemannian metric on W. Then every critical point of f has a neighborhood U in which the inequality $df(X) \geq \delta(|X|^2 + |df|^2)$ holds for some $\delta > 0$. This assertion is independent of metric.*

Note that these conditions basically makes X a pseudogradient field of f. The second means that X vanishes at critical points of f and the first means that f always changes positively along X (we view X as an operator acting on f: as "directional derivative").

A **Weinstein domain** is also a quadruple as above but (W, ω) is a compact connected symplectic manifold with boundary, f has no critical points on the boundary and in fact, $\partial W = f^{-1}(\max f)$. The Liouville vector field satisfies the two conditions above.

It can be shown that a Weinstein structure on W can always be perturbed such that f is Morse. Assuming that f is Morse, X is a pseudogradient field and so we can form stable and unstable manifolds of critical points of f using the flow of X. Let ϕ_t denote the flow of X and $p \in \operatorname{Crit} f$. Then define $\Lambda_p := \{q \in W : I_q = \mathbb{R}, \lim_{t\to\infty} \phi_t(q) = p\}$ as the stable manifold of p. Liouville vector fields enlarge the symplectic form exponentially yet Λ_p is invariant under the flow of X and in fact, the flow converges to p. This means that ω must vanish on Λ_p; i.e. Λ_p is an isotropic submanifold and hence, of dimension $\leq n = \frac{1}{2}\dim W$. This tells us that the homology of W is nontrivial only up to half the dimension. In particular, when $\dim W \geq 4$, ∂W is connected.

Definition 3.3. *A function $f : W \to \mathbb{R}$ on a complex manifold (W, J) is called **plurisubharmonic** if the 2-form $\omega_f := -d(df \circ J)$ satisfies $\omega_f(v, Jv) > 0$ for all nonzero $v \in TW$.*

Any smooth function will make ω_f of type $(1, 1)$ and so will satisfy $\omega_f(J\cdot, J\cdot) = \omega_f$. So the taming condition is enough to make ω_f compatible with J. If f is a plurisubharmonic Morse function, then the unstable manifolds are ω_f-isotropic and so its Morse critical points have indices up to middle dimension only. Moreover, if $u : \Sigma \to W$ is a J-holomorphic curve, $f \circ u$ is subharmonic and cannot have an interior maximum. This means that (W, J) does not have any nonconstant J-holomorphic curves defined on closed Riemann surfaces. I'm not sure if we really need that W is a complex manifold; can we take it to be almost complex?

Definition 3.4. *A **Stein manifold** is a connected complex manifold (W, J) without boundary that admits a plurisubharmonic function $f : W \to \mathbb{R}$ that is bounded below and proper. A **Stein domain** is a complex manifold (W, J) with boundary that admits a plurisubharmonic function f without critical points on ∂W and satisfies $\partial W = f^{-1}(\max f)$.*

One of the key properties of Stein manifolds is that they can be viewed as complex submanifolds of $\mathbb{C}^N$ that are closed as subsets. One may take $f(z) = |z|^2$ to be the plurisubharmonic function. The equivalence of these two definitions was proved by Hans Grauert in 1958. Eliashberg proved that every Weinstein manifold admits a Stein structure and every Stein manifold admits a Weinstein structure.

3.4 Affine Varieties

From the above discussion, since affine varieties are the zero sets of some finite number of polynomials in $\mathbb{C}^N$, the ones that are smooth and closed as subsets can be viewed as Stein manifolds. Indeed, in the GAGA set of analogies, Stein manifolds correspond to affine varieties.

In some of Mark's papers, he shows that all affine varieties can be compactified to Liouville domains (need to add in a contact boundary). However, not all Liouville domains admit contact boundary which themselves admit Boothby-Wang bundles and indeed, not all affine varieties admit Boothby-Wang bundles either. For example, if we remove a normal **crossing** divisor from $\mathbb{C}P^n$, as we approach the singularity, we do not have a Boothby-Wang model.

However, consider a **smooth** normal divisor D in $\mathbb{C}P^n$. It is a symplectic hypersurface and so there is a tubular neighborhood theorem which says that the normal bundle ν_D is a (real rank 2) symplectic vector bundle. Take the sphere bundle: $S(\nu_D)$. One can show that this bundle admits a Boothby-Wang structure and serves as the contact boundary of the affine variety which is formed simply by removing D. As we move towards infinity, we're moving towards D. In that situation, the S^1 fibers collapse.

3.5 Another View

Consider a subspace $W \subset (V^{2n}, \omega)$ of a symplectic vector space. The symplectic complement of W is the subspace $W^\omega := \{v \in V : \omega(v, w) = 0, \forall w \in W.\}$. W is **isotropic** if $W \subset W^\omega$, **coisotropic** if $W^\omega \subset W$, **symplectic** if $W \cap W^\omega = 0$, and **Lagrangian** if $W^\omega = W$.

One can show that coisotropic subspaces are always of at least dimension n while isotropic subspaces are at most dimension n when one uses the fact that $\dim W + \dim W^\omega = \dim V$. A submanifold $Q \subset (M, \omega)$ is given one of these adjectives if for each $q \in Q$, the subspace $T_q Q$ is that adjective in $T_p M$. It is shown in McDuff-Salamon that if Q is coisotropic, then for each $p \in Q$, $T_p Q^\omega$ is an isotropic subspace and in fact, TQ^ω is integrable. This means that coisotropic manifolds of dimension $n + k$ are foliated by isotropic leaves. When the leaves are submanifolds, they are then of dimension $n - k$. In such a situation, we let $p \sim q$ if they lie in the same leaf. Under a regularity condition, $Q/\sim$ is a smooth manifold and inherits a symplectic form from M.

One can quickly show that any hypersurface $\Sigma \in M$ is a coisotropic submanifold. If it is regular, then it has 1-dim isotropic leaves. In the case that (Σ, α) is a contact hypersurface, we have that the leaves coincide with the Reeb chords. Compare this to Boothby-Wang fibrations which do not refer to any symplectic filling of Σ. When a Boothby-Wang fibration $\Sigma \to B$ exists, there is a symplectic integral class on B. This B coincides with the $Q/\sim$ as above.

In the situation with $(M, d\lambda)$ being a Liouville domain, whenever we have a Hamiltonian H that has $\Sigma = \partial M$ as a regular level set and X_H is proportional to the Reeb vector field as described above, we have a Hamiltonian action of S^1 on Σ. Let's suppose the Reeb chords are all homeomorphic to S^1; so Σ is a regular coisotropic contact submanifold of M. Then, we have a moment map description where $\mu = H$. Let's suppose $\mu^{-1}(0) = \Sigma$. Then the symplectic quotient $\mu^{-1}(0)/S^1$ coincides with $Q/\sim$ where $Q = \Sigma$.

4 Symplectic (Co)homology

For the details of the definition of the symplectic (co)homology groups, consult Oancea's paper. I would just like to make a few remarks some of which are specific to Viterbo's setup.

Let $\widehat{M} = M \cup_{\partial M \times \{1\}} \partial M \times [1, \infty)$. Let $\widehat{\omega}$ equal ω on M, and $d(S\lambda|)$ on the rest. The gluing comes via the diffeomorphism induced by the Liouville flow. This new symplectic manifold is called the **symplectic completion** of M.

The **admissible Hamiltonians** are of the type H_μ with $\mu > 0$. $H_\mu \equiv 0$ on M and only depends time and on the radial variable S of the cylindrical end (doesn't depend on $p \in \partial M$). We would like H_μ to just be linear in S with slope μ once S is larger than 1, say, 3/2. Between 1 and 3/2, we'll like H_μ to be convex. Anyways, the H_μ are linear at infinity. However, in practice, we perturb the K_μ slightly so that $K_\mu < 0$ on $M \setminus \partial M$.

We put a partial order where $H \prec K$ if $H \leq K$ in the neighborhood of the boundary.

As usual the action functional is defined on contractible loops. If γ is such a loop, $\bar{\gamma}$ is a capping disk; we'll assume that M has the properties needed to make the action functional well-defined:

$$\mathcal{A}_H(\gamma) = -\int_{D^2} \bar{\gamma}^* \omega - \int_{S^1} H_t(\gamma(t))\, dt$$

Then, we define

$$FC^k_{(a,+\infty)}(H,J) = \bigoplus \mathbb{Z}\langle x\rangle$$

where the generators are critical points of $\mathcal{A}_H$ with Conley-Zehnder index $-k$ and action greater than a. $FH^*_{(a,+\infty)}(H,J)$ is just the cohomology and

$$FH^*_{(a,b)}(H,J) = FH^*_{(a,+\infty)}(H,J)/FH^*_{(b,+\infty)}(H,J).$$

We then define $FH^*(a,\infty)(M)$ by taking two inverse limits of (H,J) and $b\to\infty$ (the order doesn't matter here; the inverse limits commute). That is, we take Hamiltonians with steeper and steeper slope and bigger and bigger range of action.

Note that when $a<0$, the Floer homology is independent of a. This is because our cofinal family of Hamiltonians, when we perturb them slightly will be slightly negative on M, say $-\delta$ for small $\delta>0$. But in the inverse limit, we may take $\delta\to 0$. So then the action of a constant orbits, say γ_0, depends on $\int_0^1 H_t(\gamma_0)\,dt \to 0$.

One important thing to note is that unlike the closed case, we need a C^0 bound which ensures that for fixed limiting orbits, the Floer trajectories stay in a compact set. If this weren't the case, we could have that $\partial^2\neq 0$ or other pathological situations.

Another note is that we could have defined this for homology. The direct limit is an exact functor while inverse limit is only left exact. Thus, if we use homology, the Künneth formula is valid with any coefficients but in cohomology, it holds only for fields.

Suppose K_μ is a Hamiltonian. Note that the 1-periodic orbits of K_μ are the constant orbits on the interior of M and the closed characteristics on ∂M with action at most μ. How does one obtain the C^0 bounds which keep the Floer trajectories within a compact set?

We note that for generic μ, there are no characteristics with period μ and thus all the 1-periodic orbits of K_μ are in a neighborhood of M. To see why, recall that on $\partial M\times[1,\infty)$, $X_{K_\mu} = -k'(S)X_R$ where $K_\mu\equiv k(S)$ on the cylindrical end and when $S\gg 1$, $k'(S)=\mu$. So for $S\gg 1$, 1-periodic orbits of X_{K_μ} are precisely the μ-periodic orbits of X_R. For generic μ, there are no such orbits. And so all the 1-periodic nonconstant orbits must be found in $\partial M\times[1,\sigma)$ where σ is the first point where $k'(S)=\mu$. So all orbits of X_{K_μ} are found in this neighborhood of M: $M\cup\partial M\times[1,\sigma)$.

Thus, if a Floer trajectory connecting 1-periodic orbits were to leave the neighborhood, it would have to loop back and thus, have a interior tangency with some slice $\partial M\times S_0$. But this phenomenon is forbidden, by a **maximum principle** argument (see p. 14 of the paper). This relies on the Liouville field being outward pointing.

Let's demonstrate this first with a toy model. We begin with a definition:

Definition 4.1. *Let J be an almost complex structure compatible with the symplectic form ω. A hypersurface $\Sigma\subset M$ is **J-convex** if it can be locally written as the regular level set of a **plurisubharmonic function**; i.e. a function $\varphi: M\to\mathbb{R}$ which satisfies $dd^c\varphi<0$ where $d^c = J^*d$.*

Remark . *Compare the definition for plurisubharmonic here and from before.*

Lemma 4.2. *Let $\Sigma\subset M$ be a J-convex hypersurface and φ a local function of definition. Then no J-holomorphic curve $u:(D^2,i)\to M$ can have an interior strict tangency point with Σ; i.e. $\varphi\circ u$ cannot have a strict local maximum.*

Proof. Here's the simple argument. Let $z=s+it$ be complex coordinates of D^2 and $J_0=i$, the standard complex structure on the disk. Let $f=\varphi\circ u$. We note that $d^cf = df\circ J_0$. Now,

$$df = \frac{\partial f}{\partial s}ds + \frac{\partial f}{\partial t}dt.$$

$df(J\partial_s) = df(\partial_t) = \partial f/\partial t$ and $df(J\partial_t) = -df(\partial_s) = -\partial f/\partial s$. From this, we can deduce that

$$d^c f = \frac{\partial f}{\partial t} ds - \frac{\partial f}{\partial s} dt \text{ and } dd^c f = \frac{\partial^2 f}{\partial t^2} ds \wedge dt - \frac{\partial^2 f}{\partial s^2} dt \wedge ds = -\Delta(\varphi \circ u) ds \wedge dt.$$

More over, the J-holomorphicity of u implies that $dd^c_{J_0}(\varphi \circ u) = dJ_0^* u^* d\varphi = du^* J^* d\varphi = u^* dd^c_{J_0}\varphi$. Since φ is plurisubharmonic, then $\Delta(\varphi \circ u) \geq 0$. Then $f = \varphi \circ u$ satisfies the mean value inequality which means it cannot obtain a strict interior maximum.

$$f(z_0) \leq \frac{1}{\pi r^2} \int_{\partial B(z_0, r)} f(z)\, dz.$$

□

Then, here is the more relevant lemma to Floer theory.

Lemma 4.3. *Any solution $u : (D^2(0,1), i) \to \partial M \times [1-\epsilon, \infty)$ of the Floer equation:*

$$u_s + Ju_t - \nabla H(s, t, u(s,t)) = 0$$

with J a standard almost complex structure, $H(s,t,p,S) = h(s,t,S)$ and $\frac{\partial^2 h}{\partial s \partial S} \geq 0$ cannot have a strict interior tangency with some slice $\partial M \times \{S_0\}$.

5 (Wrapped) Lagrangian Floer Homology

Much of this is from McLean's paper *Affine Varieties, Singularities and the Growth Rate of Wrapped Floer Cohomology*. Let's consider Liouville domains (M, ω) where the boundary is our contact type hypersurface. We will try to be consistent with notation, letting $\lambda = \iota_X \omega$ and $\theta = \lambda|_{\partial M}$. Let $\widehat{M}$ be the completion and S be the coordinate for $[1, \infty)$ on the cylindrical end.

Definition 5.1. *A (possibly non-compact) properly embedded submanifold of $L \subset \widehat{M}$ is said to be an **exact Lagrangian which is cylindrical outside** M if*

- *it is of half the dimension of $\widehat{M}$,*
- *$\theta|_L = df_L$ for some smooth $f_L : L \to \mathbb{R}$ where $f_L = 0$ outside M*
- *the vector field ∂_s is tangent to L in the cylindrical end $[1, \infty) \times \partial M$.*

*We say that f_L is a function **associated** to L. An **admissible Lagrangian** is an oriented exact Lagrangian which is cylindrical outside M with a choice of spin structure.*

An example of an exact Lagrangian which is cylindrical outside M may look like products of Legendrians with $\mathbb{R}$ on the cylindrical end. Since Legendrians are transverse to the Reeb vector field, they are transverse to the fibers of the Boothby-Wang fibers. Let's take the definition of admissible Hamiltonian H to mean *linear outside a compact set containing M*. Inside, of M, H could behave in any number of ways. We then see that the orbits of X_H outside of M are proportional to the orbits of X_R and so on the cylindrical end, the Lagrangians are also transverse to any admissible Hamiltonian vector field.

5.1 The Action Functional

Normally, when defining Lagrangian Floer homology $HF_*(L_0, L_1)$ for two Lagrangians L_0, L_1, we form a chain complex using intersection points in $L_0 \cap L_1$ where we require the intersection to be transverse. The differential is then defined by counting J-holomorphic strips u; that is, solutions of $\bar{\partial}_J u = 0$.

However, we often choose to perturb our situation by a (time-dependent) Hamiltonian $H_t : S^1 \times \widehat{M} \to \mathbb{R}$ because, for example, L_0, L_1 might not intersect transversally but $\phi_1^{H_t}(L_0)$ and L_2 do intersect transversally. On the cylindrical end, for example, if H_t is admissible, then taking $L = L_0 = L_1$, we can consider intersection points $\phi_1^{H_t}(L) \cap L$. When H_t is admissible, it is also rather inaccurate to call it a perturbation since these are not small changes but large changes.

Remark . *We will not always be careful about including the variable t but recall that we are considering time dependent Hamiltonians.*

With that said, let L_0, L_1 be two admissible Lagrangians with associated functions f_0, f_1. Let $p \in \phi_1^H(L_0) \cap L_1$. Let $\gamma(t) = \phi_t^H(\phi_{-1}^H(p))$ be a path that starts on L_0 and ends at $\phi_1^H(L_0)$. Here, I've written ϕ_{-1}^H to mean $(\phi_1^H)^{-1}$. Note that $\gamma(1) = p$. Then the action of p is given by

$$\mathcal{A}_H(p) = f_1(\gamma(1)) - f_0(\gamma(0)) - \int \gamma^*\theta + \int_0^1 H_t(\gamma(t))\, dt.$$

Note that if we define $\mathcal{A}_H$ on paths ending at intersection points p, then these intersection points are in 1-1 correspondence with critical paths of $\mathcal{A}_H$. When looking at the cylindrical end, these paths are what we've called Reeb chords above.

We may obtain a chain complex $C^*_{[a,b]}(L_0, L_1, H_t)$ which is the free $\mathbb{K}$ vector space generated by intersection points $\phi_1^H(L_0) \cap L_1$ whose action is in $[a, b]$. If we do not write $[a, b]$, what we mean is that $a = -\infty, b = +\infty$.

It is possible to assign indices to the intersection points: $|p| \in \mathbb{Z}_2$; this makes $C^*_{[a,b]}(L_0, L_1, H_t)$ a $\mathbb{Z}_2$ graded $\mathbb{K}$ vector space. We could go into more detail with gradings but we do not have much need for them here. Suffice to say, the spin structures on the Lagrangians are quite important though.

5.2 The Floer Equation and Differential

To define the differential, we need to consider **cylindrical almost complex structures**. These are simply complex structures J compatible with ω and outside a large compact set, $\theta \circ J = dS$. So for example, $dS(\partial_S) = 1 = \theta(X_R)$. So we need $J\partial_S = X_R$. For all vector fields $V \in \ker\theta$, we have $dS(V) = 0 = \theta(JV)$. So J leaves $\ker\theta$ invariant.

Let J_t be a smooth S^1 family of cylindrical ACS. We define $\mathcal{M}(p, q, H_t, J_t)$ to be the set of smooth maps $u : \mathbb{R}_s \times [0, 1]_t \to M$ satisfying:

$$\bar{\partial}_{J_t} u := \partial_s u + J_t \partial_t u = J_t X_{H_t},$$
$$u(s, 0) \in L_0, u(s, 1) \in L_1,$$
$$u(s, t) \to p(t) \text{ as } s \to -\infty \text{ and } u(s, t) \to q(t) \text{ as } s \to +\infty.$$

Alternatively, we could have u satisfy $\bar{\partial}_{J_t} u = 0$ and have boundary conditions $u(s, 0) \in \phi_1^H(L_0), u(s, 1) \in L_1$ instead. As usual, there is a free $\mathbb{R}$ action on $\mathcal{M}(p, q, H_t, J_t)$. The spin structures guarantee orientations on $\overline{\mathcal{M}}(p, q, H_t, J_t) = \mathcal{M}(p, q, H_t, J_t)/\mathbb{R}$. It can be shown that when J_t is generic and $|p| = |q| + 1$, there is a stratification by j, the dimension:

$$\overline{\mathcal{M}}(p, q, H_t, J_t) = \bigsqcup_j \overline{\mathcal{M}}^j(p, q, H_t, J_t)$$

We define the differential of the chain complex by considering only $\overline{\mathcal{M}}^0(p, q, H_t, J_t)$:

$$\partial(q) := \sum_{|p|=|q|+1} \#\overline{\mathcal{M}}^0(p, q, H_t, J_t) \cdot p.$$

Claim: The differential increases the action and so the chain complex has a natural filtration given by $p \mapsto \mathcal{A}_H(p)$. Moreover, $\partial^2 = 0$.

As usual, we let $HF^*_{[a,b]}(L_0, L_1, H_t)$ be the homology of the above chain complex. If H^1_t, H^2_t are two admissible Hamiltonians with $H^1_t \leq H^2_t$ for all $t \in S^1$, then we can define **continuation maps** $HF^*_{[a,b]}(L_0, L_1, H^1_t) \to HF^*_{[a,b]}(L_0, L_1, H^2_t)$.

These are constructed in the usual way. We take a homotopy (a path in the space of Hamiltonians and ACS pairs) $\Gamma(s) = (H_{s,t}, J_{s,t})$ which connects (H^1_t, J^1_t) to (H^2_t, J^2_t). Define a chain map $\Phi^\Gamma : C^*_{[a,b]}(L_0, L_1, H^1_t, J^1_t) \to C^*_{[a,b]}(L_0, L_1, H^1_t, J^1_t)$ which is defined by counting solutions to a newly parametrized Floer equation:

$$\partial_s u + J_{s,t}\partial_t u = J_{s,t} X_{H_{s,t}}$$

It can be shown that $\Phi^\Gamma \circ \partial^1 = \partial^2 \circ \Phi^\Gamma$ and so it descends to a map on homology. Moreover, it can be shown that the choice of homotopy does not matter (so long as we pass through generic J_t). Thus, I think we can take a "straight line" homotopy between H^1 and H^2 in the sense that since they're linear at infinity, we can just change their slope. Of course, more needs to be done in a neighborhood of M.

Now suppose that $\phi^H_1(L_0)$ does not intersect L_1 transversally and that a, b are not in the image of $\mathcal{A}_H$. Then we define $HF^*_{[a,b]}(L_0, L_1, Ht)$ to be the direct limit

$$\varinjlim_{H'_t} HF^*_{[a,b]}(L_0, L_1, H'_t)$$

where $H'_t < H_t$ are admissible Hamiltonians so that $\phi^{H'}_1(L_0)$ and L_1 intersect transversally and the directed system is taken with respect to the ordering $\leq$. Sometimes one cannot find such admissible Hamiltonians H'_t which C^∞ converge to H_t. In this case one needs to have more general Hamiltonians.

5.3 Two Views

It bears repeating that there are at least two equivalent views for thinking about Lagrangian Floer homology. The generators of the chain complex can either be thought of as intersection points of $\phi^H_1(L_0) \cap L_1$ or critical paths of the action functional $\mathcal{A}_H$ (which are called Reeb chords).

To define the differential of the chain complex, we study strips. We can either consider strips u that satisfy $\bar{\partial}_J u = 0$ with boundary conditions on $\phi^H_1(L_0)$ and L_1 or strips u that satisfy $\bar{\partial}_J u = JX_H$ with boundary conditions on L_0 and L_1.

5.4 Properties of HF

1. The rank of $HF^*_{[a,b]}(L_0, L_1, H_t)$ is a lower bound on the the number of intersection points of $\phi^H_1(L_0) \cap L_1$ whose actions are in $[a, b]$ when the intersections are transverse.

2. If $a_1 \geq a_2$ and $b_1 \geq b_2$ then there is a natural morphism:

$$HF^*_{[a_1,b_1]}(L_0, L_1, H_t) \to HF^*_{[a_2,b_2]}(L_0, L_1, H_t).$$

We call such a morphism an **action morphism**. Composing two action morphisms gives another action morphism. Similar properties hold for other intervals of the form $[a, b), (a, b]and(a, b)$. Such a morphism is an isomorphism if there are no intersection points $p \in \phi_1^H(L_0) \cap L_1$ of action $\mathcal{A}_H(p)$ in the interval $[a_2, b_1] \setminus [a_1, b_2]$. For $-\infty \leq a \leq b \leq c \leq +\infty$ we have the following long exact sequence:

$$\to HF^*_{(b,c]}(L_0, L_1, H_t) \to HF^*_{[a,c]}(L_0, L_1, H_t) \to HF^*_{[a,b]}(L_0, L_1, H_t) \to$$

where the morphisms which aren't connecting morphisms are action morphisms.

3. If $H_t^1 \leq H_t^2$, then there is a natural morphism $HF^*_{[a,b]}(H_t^1) \to HF^*_{[a,b]}(H_t^2)$. This is a **continuation morphism** and the composition of two such morphisms is also a continuation morphism.

4. Continuation morphisms commute with action morphisms (need to arrange the intervals and Hamiltonians).

5. Let $c \in \mathbb{R}$ be a constant. Then we have an isomorphism $HF^*(L_0, L_1, H_t) \to HF^*(L_0, L_1, H_t + c)$. If $c > 0$, the morphism is induced by the natural contiuation map. If $c < 0$, then it is induced by the inverse of a continuation map.

5.5 Wrapped Lagrangian Floer Homology

We let $H \geq 0$ be an admissible Hamiltonian with positive slope. By property 3, we have natural continuation maps $HF^*(L_0, L_1, \lambda_1 H) \to HF^*(L_0, L_1, \lambda_2 H)$ for $\lambda_1 \leq \lambda_2$.

Definition 5.2. *Define the wrapped Floer cohomology group of L_0 and L_1 to be*

$$HW^*(L_0, L_1, H, \mathbb{K}) := \varinjlim_{\lambda} HF^*(L_0, L_1, \lambda H).$$