

# Math 234; Ch. 6: Laplace Transforms

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These are notes I originally wrote up for myself so not everything is intended for students though most things are.

## 1 Section 6.1: Laplace Transforms

So far, we've been studying some of the behavior of ODEs via qualitative methods. For example, we find the eigenvalues associated to an equilibrium point and can then describe the equilibrium point and the nearby solutions; e.g. a spiral sink. However, we have not written down explicit analytic solutions for most of the nonlinear examples. We will now discuss a powerful tool that aids us in finding analytic solutions.

To start, integral transforms come up in many application of mathematics, engineering, physics, etc. For instance, given a function  $x(t)$ , the “amount” of the frequency  $\omega$  in the function  $x$  can be described by the two real integrals

$$\int_{-\infty}^{\infty} x(t) \cos(\omega t) dt \text{ and } \int_{-\infty}^{\infty} x(t) \sin(\omega t) dt$$

or by one complex integral; this is the Fourier transform. In differential equations we have seen that exponentials play a huge role in some of the 1st and 2nd order ODEs. So we may similarly ask: how much do exponentials make up a given function  $y(t)$ ?

**Definition 1.1.** The **Laplace transform** of an integrable function  $y : [0, \infty) \rightarrow \mathbb{R}$  is given by the integral transform

$$\mathcal{L}[y](s) = \int_0^{\infty} y(t) e^{-st} dt.$$

The functions you're used to are those where we input a number and get out a number. Here, we input a function and get out a function. So  $\mathcal{L}$  is called an **operator** where we put in  $y(t)$  and get out another function  $Y(s)$ ; note the change in variable! We are in a new domain because the integration is in  $t$  so we get rid of all the  $t$ 's in this fashion. We expect that  $\mathcal{L}[y]$  is once-differentiable even if  $y$  is not. Before doing an example, we know that integration is linear. So we immediately have two properties:

**Proposition 1.2.** Let  $f, g : [0, \infty) \rightarrow \mathbb{R}$  be two integrable functions and  $\alpha \in \mathbb{R}$ . Then  $\mathcal{L}[f+g] = \mathcal{L}[f] + \mathcal{L}[g]$  and  $\mathcal{L}[\alpha f] = \alpha \mathcal{L}[f]$ . In particular,  $\mathcal{L}[0] = 0$ .

That is,  $\mathcal{L}$  is a **linear** operator (it's important that it preserves 0). Matrices are examples of linear operators as well but the ones we've seen are for finite dimensions.  $\mathcal{L}$  acts on a space of integrable functions so it is an  $\infty$ -dimensional operator.

**Example 1.3.** Let  $a \in \mathbb{R}$ . Then

$$\mathcal{L}[e^{at}](s) = \int_0^{\infty} e^{at} e^{-st} dt = \int_0^{\infty} e^{(a-s)t} dt = \left. \frac{e^{(a-s)t}}{a-s} \right|_0^{\infty} = \begin{cases} \infty & s \leq a \\ \frac{1}{s-a} & s > a \end{cases}.$$

Since this diverges for  $s \leq a$ , we'll take the part that we can use which is when  $s > a$ . Also, from this, by writing  $1 = e^{0 \cdot t}$ , we immediately see that  $\mathcal{L}[\alpha] = \alpha \mathcal{L}[1] = \frac{\alpha}{s}$ . Now, why are these useful? It seems difficult to compute some of these integrals. But here's a proposition.

**Proposition 1.4.** *Given a function  $y(t)$  with slow growth at  $+\infty$ ; i.e.  $\lim_{t \rightarrow +\infty} y(t)e^{-st} = 0$ , we have*

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0).$$

*Proof.* Using Integration By Parts,

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \int_0^\infty \frac{dy}{dt} e^{-st} dt = y(t)e^{-st} \Big|_0^\infty - \int_0^\infty -sy(t)e^{-st} dt.$$

For the 2nd term, since we're not integrating in  $s$ , we can treat it as a variable and move it out. Then we just have the definition of the Laplace transform of  $y$ . So the 2nd term (once the minus signs cancel), is  $s\mathcal{L}[y]$ . As for the first, our assumption is that the limit is 0 so we're just left with evaluating  $-y(0)$ . Hence, we have the result.  $\square$

So note that we traded a derivative in the  $t$ -domain for multiplication by  $s$  in the  $s$ -domain. This looks useful because derivatives can be hard to compute but it's easy to multiply by  $s$ ; that's just algebra, not calculus. Before we see how to use these for differential equations, we need one more result.

**Theorem 1.5.** *The Laplace transform is unique. That is, if  $Y(s) = \mathcal{L}[y](s)$  and also  $Y(s) = \mathcal{L}[x](s)$ , then  $y(t) = x(t)$ . Because of this, then  $\mathcal{L}$  is invertible and it makes sense to define  $\mathcal{L}^{-1}$ . In particular,  $\mathcal{L}^{-1}[Y](t) = y(t)$ .*

The proof is not too hard and I'll put it in the Appendix. Now for an example.

**Example 1.6.** Consider the initial value problem (IVP)  $y' = y + 2e^{-t}$  with  $y(0) = 1$ . We may apply the Laplace transforms to both sides to get  $\mathcal{L}[y'] = \mathcal{L}[y] + 2\mathcal{L}[e^{-t}]$ . The LHS can be replaced with  $s\mathcal{L}[y] - 1$  using our proposition and the fact that  $y(0) = 1$ . Rearranging our equation, we get

$$(s-1)\mathcal{L}[y] = 1 + 2\mathcal{L}[e^{-t}] \longrightarrow \mathcal{L}[y] = \frac{1}{s-1} + \frac{2}{(s-1)(s+1)}.$$

Here, we got the last term because earlier, we had computed that  $\mathcal{L}[e^{at}] = 1/(s-a)$ . So here,  $a = -1$ . Now, we need to apply the inverse Laplace transform and we'll have found a solution  $y(t)$  to the IVP. One trouble is dealing with the product of two fractions. However, we have the method of partial fractions decomposition from Calculus II. As a reminder, it goes like this:

$$\frac{2}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} \longrightarrow 2 = A(s+1) + B(s-1).$$

This means that  $A+B=0, A-B=2 \rightarrow A=1, B=-1$ . Plugging in, we have

$$\mathcal{L}[y] = \frac{1}{s-1} + \frac{1}{s-1} - \frac{1}{s+1} \rightarrow y(t) = 2\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = 2e^t - e^{-t}.$$

As a check,  $y(0) = 2-1 = 1$  and  $y'(t) = 2e^t + e^{-t}$ . On the other hand,  $y + 2e^{-t} = 2e^t - e^{-t} + 2e^{-t} = 2e^t + e^{-t}$ . So this  $y$  is a solution to the IVP.

## 2 Section 6.2: Discontinuous Functions

Laplace transforms can also help us solve ODEs that we have been unable to address so far. Namely those with discontinuous functions appearing. As a warmup, recall that if  $y(t) = |t|$ , then its derivative is not defined at 0 but elsewhere, its derivative is  $-1$  for  $t < 0$  and  $+1$  for  $t > 0$ . The **step** (or Heaviside) function at  $a \in \mathbb{R}$  is

$$u_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a. \end{cases}$$

This may show up, for example, in a model of a forced harmonic oscillator where the forcing term is  $7u_3(t) \sin(16t)$ . This means that there is no forcing for the first 3 seconds and then it is suddenly “turned on” to be  $7 \sin(16t)$ . This sort of thing would be relevant if, for example, you had a circuit board and you switched on some current.

To make use of  $u_a$ , let's find its Laplace transform. We'll usually assume  $a > 0$ .

$$\mathcal{L}[u_a] = \int_0^\infty u_a(t)e^{-st} dt = \int_a^\infty e^{-st} dt = \frac{e^{-st}}{-s} \Big|_a^\infty = \frac{e^{-as}}{s}.$$

So we now know the inverse Laplace transform of  $e^{-as}/s$ ; it's  $u_a(t)$ . We can get even more sophisticated.

**Proposition 2.1.**  $\mathcal{L}[u_b(t)e^{a(t-b)}] = \frac{e^{-bs}}{s-a}$ .

You'll prove this in the homework (it's an easy computation). This then tells us something about inverse Laplace transforms as well which we'll use in the next example.

**Example 2.2.** Consider the following IVP where  $y(0) = 5$  and

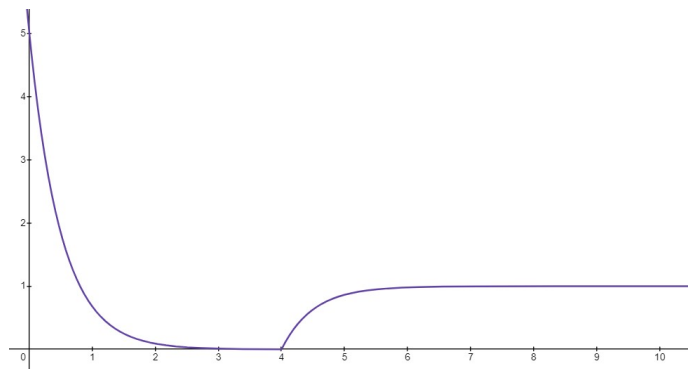
$$y'(t) = \begin{cases} -2y & t < 4 \\ -2y + 2 & t \geq 4. \end{cases}$$

This means that we suddenly get a change in  $y'(t)$  by 2 at time  $t = 4$ . In other words, we can rewrite the ODE as  $y' = -2y + 2u_4(t)$ . Applying the Laplace transform to both sides, we have

$$s\mathcal{L}[y] - y(0) = -2\mathcal{L}[y] + 2\mathcal{L}[u_4] \rightarrow (s+2)\mathcal{L}[y] = 5 - 2\mathcal{L}[u_4] \rightarrow \mathcal{L}[y] = \frac{5}{s+2} - 2\frac{e^{-4s}}{s(s+2)}.$$

We want to apply the inverse Laplace transform. But let's first use partial fractions on  $\frac{2}{s(s+2)} = \frac{1}{s} - \frac{1}{s+2}$ . Hence, the 2nd term above splits into  $\frac{e^{-4s}}{s} - \frac{e^{-4s}}{s+2}$ .

Substituting in and applying the inverse Laplace transform and also the proposition above which you'll prove in homework, we have  $y(t) = 5e^{-2t} + u_4(t) - u_4(t)e^{-2(t-4)}$ . Here's a graph of the solution; note that it satisfies the initial condition and at time  $t = 4$ , the solution is nondifferentiable because  $u_4$  is turned on suddenly. Compare this to  $A(x) = \int_0^x u_a(t) dt$  which is also continuous but not differentiable.



Note that if we want to “turn off” a function at time  $a$ , we may use  $u_0(-t - a)$ . First,  $u_0(-t)$  reflects the step function across the  $y$ -axis and then the  $-a$  translates.

### 3 Section 6.3: 2nd Order ODEs

We can use Laplace transforms to solve 2nd order ODEs as well (and higher ones). Note that by using Proposition 1.4 repeatedly, we can work with  $n$  derivatives. For  $n = 2$ ,  $\mathcal{L}[y''] = s\mathcal{L}[y'] - y'(0) = s(s\mathcal{L}[y] - y(0)) - y'(0) = s^2\mathcal{L}[y] - sy(0) - y'(0)$ . So we're trading the number of derivatives for powers of  $s$ . For  $n = 3$ , we'll have  $s^3\mathcal{L}[y] - s^2y(0) - sy'(0) - y''(0)$ .

Before working on 2nd order ODEs, let's find the Laplace transforms for some sinusoidal functions. If we use a complex exponential  $e^{i\omega t}$ , then

$$\mathcal{L}[e^{i\omega t}] = \int_0^\infty e^{(i\omega - s)t} dt = \frac{1}{s - i\omega} = \frac{1}{s - i\omega} \frac{s + i\omega}{s + i\omega} = \frac{s + i\omega}{s^2 + \omega^2}.$$

Taking real and imaginary part, we have

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2 + \omega^2}; \quad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2}.$$

In fact, let's go a bit further:

$$\mathcal{L}[e^{(a+i\omega)t}] = \frac{1}{(s-a) - i\omega} = \frac{1}{(s-a) - i\omega} \frac{(s-a) + i\omega}{(s-a) + i\omega} = \frac{(s-a) + i\omega}{(s-a)^2 + \omega^2}.$$

So

$$\mathcal{L}[e^{at} \cos(\omega t)] = \frac{s-a}{(s-a)^2 + \omega^2}; \quad \mathcal{L}[e^{at} \sin(\omega t)] = \frac{\omega}{(s-a)^2 + \omega^2}.$$

**Example 3.1.** Consider the IVP  $y'' + 4y = 0$ ,  $y(0) = y_0$ ,  $y'(0) = v_0$ . Then, taking the Laplace transform, we have

$$\mathcal{L}[y''] + 4\mathcal{L}[y] = 0 \longrightarrow s^2\mathcal{L}[y] + 4\mathcal{L}[y] - sy(0) - y'(0) \longrightarrow (s^2 + 4)\mathcal{L}[y] = sy_0 + v_0.$$

Hence,

$$\mathcal{L}[y] = \frac{y_0 s}{s^2 + 4} + \frac{v_0}{s^2 + 4}.$$

We'd like to apply the inverse Laplace transform. Since  $y_0$  and  $v_0$  are constants, we can use linearity to deal with them. The first one is then going to be  $\cos(2t)$  but for the 2nd, we need to introduce a 2 in the numerator in order to use our computation of  $\mathcal{L}[\sin(\omega t)]$ . So just artificially, we factor out  $v_0/2$ . Then, we can use  $\mathcal{L}^{-1}$  and get  $y(t) = y_0 \cos(2t) + \frac{v_0}{2} \sin(2t)$ .

Notice by the way that if we have the general harmonic oscillator without forcing, the equation is  $y'' + py' + qy = 0$  and the Laplace transform of it is  $(s^2 + ps + q)\mathcal{L}[y] - sy(0) - y'(0) = 0$ . Note that we have the characteristic polynomial on the left, just with  $\lambda$  replaced by  $s$ ! So the Laplace transform verifies that the characteristic polynomial is useful for harmonic oscillators.

**Example 3.2.** Let's add some forcing terms to the previous initial value problem:  $y'' + 4y = 3\sin t$ ,  $y(0) = y_0$ ,  $y'(0) = v_0$ . We can make use of what we did above but instead of setting something to 0, we set it to  $3\mathcal{L}[\sin t]$ .

$$s^2\mathcal{L}[y] + 4\mathcal{L}[y] - sy(0) - y'(0) = 3\mathcal{L}[\sin t] = \frac{3}{s^2 + 1} \rightarrow \mathcal{L}[y] = \frac{y_0 s}{s^2 + 4} + \frac{v_0}{s^2 + 4} + \frac{3}{(s^2 + 4)(s^2 + 1)}.$$

We already know the inverse Laplace transforms for the first two so we don't need to do anymore work. Note that the linearity of Laplace transforms actually proves for us that having found homogeneous solutions in Example 3.1, we only need to find a particular solution. For us, this means applying  $\mathcal{L}^{-1}$  to the final term. But to do that, we need a slightly more sophisticated partial fractions decomposition. Here is a reminder:

**Lemma 3.3.** *A polynomial with real coefficients can always be factored into a product of linear terms and irreducible quadratic terms. An irreducible quadratic is one whose roots are not real but have nontrivial imaginary part. Moreover, when doing partial fractions decomposition, whenever we see  $(ax^2 + bx + c)^k$  in the denominator where  $ax^2 + bx + c$  is irreducible, then our partial fractions decomp will have terms*

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}.$$

**Remark .** *This lemma is implied by the fact that  $\mathbb{C}$  is a (Galois) field extension over  $\mathbb{R}$  of degree 2; the canonical generator for the Galois group is complex conjugation. More to the point,  $\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1)$  as  $\mathbb{R}$ -algebras and is one of simplest Clifford algebras. All algebras can be described as quotients of polynomial rings by ideals.*

In our example, both  $s^2 + 1$  and  $s^2 + 4$  are irreducible because their roots are  $\pm i, \pm 2i$ , respectively. Doing PFD, we get  $\frac{3}{(s^2+4)(s^2+1)} = \frac{1}{s^2+1} - \frac{1}{s^2+4}$ . The inverse Laplace transform of the first is  $\sin t$  and for the second, it's  $-\frac{1}{2} \sin(2t)$ . Hence,  $y(t) = y_0 \cos(2t) + \frac{v_0}{2} \sin(2t) + \sin t - \frac{1}{2} \sin(2t)$ .

**Example 3.4.** Here is a damped harmonic oscillator (no forcing).  $x'' + 2x' + 5x = 0$ ,  $x(0) = 4$ ,  $x'(0) = 0$ . By linearity and using the fact that derivatives turn into powers of  $s$  under  $\mathcal{L}$ ,

$$\mathcal{L}[x'' + 2x' + 5x] = \mathcal{L}[0] = 0 \longrightarrow (s^2 + 2s + 5)\mathcal{L}[y] = 4s + 8.$$

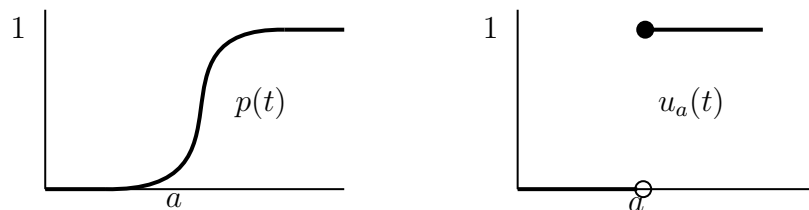
To make use of our computation above, we need to do some completing the square:  $s^2 + 2s + 5 = (s + 1)^2 + 4$ . We'll also write  $4s + 8 = 4(s + 1) + 2 \cdot 2$ . Then everything is in place for applying inverse Laplace transform to the following expression.

$$\mathcal{L}[x] = \frac{4(s + 1)}{(s + 1)^2 + 4} + 2 \frac{2}{(s + 1)^2 + 4}.$$

We find that  $x(t) = 2e^{-t}(2 \cos(2t) + \sin(2t))$ . A quick check shows that the initial conditions are satisfied.

## 4 Section 6.4: Delta Functions and Impulse Forcing

Consider the graphs of functions below; on the left, we have a smooth function which is 0 for  $t$  much smaller than a fixed number  $a$  and is 1 for values much larger than  $a$ . Around  $t = a$ , the function grows rapidly (has large derivative) but away from  $t = a$ , the derivative is mostly equal to 0. If we imagine making the region on which the function grows to be smaller and smaller, then the limit would be for the function to become discontinuous, jumping from 0 to 1 immediately at  $t = a$ . That is, we get a step function, shown on the right in the figure.

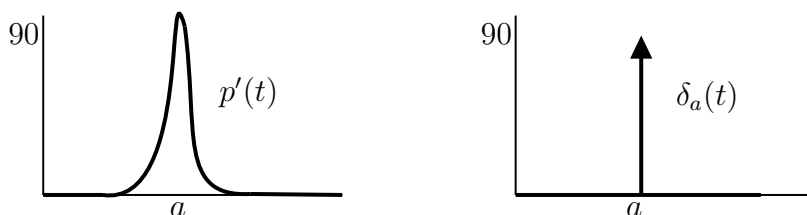


Recall the step function

$$u_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a. \end{cases}$$

The point of introducing these two functions is to discuss the notion of “impulse” forcing. Imagine that the function on the left, call it  $p$ , represents momentum of an object in space (no friction). For most of the time before  $t = a$ , the object is at rest but then something exerts force on it in a brief timeframe and it gains some momentum. If the duration in which the force is exerted is very small, a small pulse, then this kind of force is called **impulse**. e.g. the length of time in which a baseball bat makes contact with a baseball is quite small but it changes the baseball’s velocity (and hence, momentum) dramatically. Or say you have a spring oscillating the back and forth; as it goes back, you set up a barrier so that as it comes forward, it suddenly crashes into the barrier and can go no further (though it can bounce back). This amounts to the wall exerting a force on the spring and there’s a discontinuity in the velocity since it suddenly jumps.

Mathematically, in the limit,  $u_a$  should represent an instantaneous impulse: the force is imparted in an instant. This is an idealization. Since the derivative of momentum is force, we’d like to be able to differentiate  $u_a$  at  $a$ . Of course, the function  $p$  is differentiable everywhere and the graph of its derivative might look like the example below. But  $u_a$  isn’t differentiable at  $a$ ; it’s not even continuous so there isn’t actually an honest-to-goodness function which is the derivative of  $u_a$ . The limits of the graphs would look something like what’s shown below on the right but it clearly fails the vertical line test and so isn’t really a function. I put 90 as a value just to show that the 1st derivative could be rather large.



However, we might want to work with this “idealized” situation of instantaneous impulse. For one, it can be difficult to explicitly write down functions like  $v$  even though it’s easy to draw them and describe the properties they ought to have. Motivated by this, let’s imagine there were such a function that’s the derivative of  $u_a$ , call it  $\delta_a$  which, away from  $t = a$ , should equal 0. Integration by parts of functions says  $\int f g' = f g - \int f' g$  so if we multiply  $\delta_a$  with any differentiable function  $f$  that takes on nonzero values only on some interval  $[-M, M]$ , in this fantasy, we’d like

$$\int_{-\infty}^{\infty} f(t) \delta_a(t) dt = f(t) u_a(t) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t) u_a(t) dt.$$

Since  $f$  equals 0 at  $\pm\infty$ , the first term vanishes. Because  $u_a(t) = 0$  for  $t < a$ , 1 for  $t \geq a$ , the second term equals

$$- \int_a^{\infty} f'(t) dt = f(a).$$

So integrating such functions  $f$  (called compactly supported functions) against  $\delta_a$  on  $\mathbb{R}$  is the same thing as **evaluating**  $f$  at  $t = a$ . For convenience, let’s suppose  $a > 0$ . So this will be the defining property of our “function”: integrating any compactly supported differentiable  $f$  against  $\delta_a$  returns  $f(a)$ . It’s worth making a bit of a philosophical point here: in math, we have the freedom to create new ideas and instead of trying to say what something **is** (since there are no functions with the property we want), we describe what something **does**. This  $\delta_a(t)$  is called the **Dirac delta function** and is the next best thing we have to a “derivative” of the step function. As we’ve said, it’s not really a function; the nomenclature is an artifact

of history and it's more accurately a so-called tempered distribution because we may use it to define the evaluation **functional**  $ev[f] := \int_0^\infty \delta_a(t)f(t) dt = f(a)$ .

With this in hand, the Laplace transform of  $\delta_a$  is just evaluation. Strictly speaking,  $e^{-st}$  isn't compactly supported but it limits to 0 for  $t \rightarrow +\infty$ . We don't need to deal with  $-\infty$  since  $u_a = 0$  there anyways.

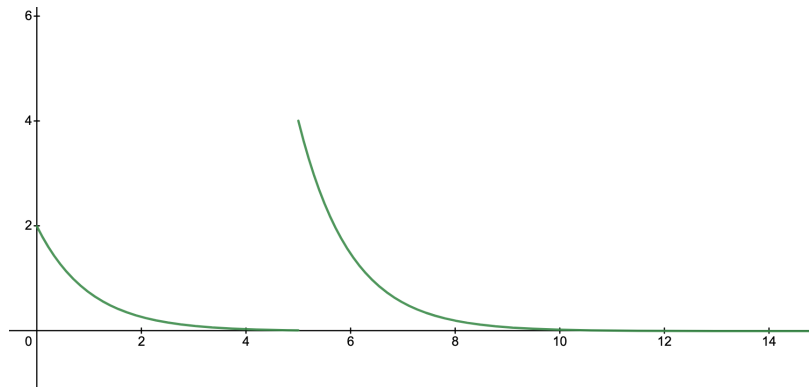
$$\mathcal{L}[\delta_a] = \int_0^\infty \delta_a(t)e^{-st} dt = e^{-sa}.$$

Recall that we take  $s > 0$  as just a variable to use. Anyways, let's apply these ideas to an initial value problem.

**Example 4.1.**  $\frac{dy}{dt} = -y + 4\delta_5(t), y(0) = 2$ . At time  $t = 5$ , the velocity gets a sudden "kick." Applying the Laplace transform to both sides and using linearity (see below for a bit more detail on the computations):

$$\begin{aligned} \mathcal{L}[\dot{y}] &= \mathcal{L}[-y + 4\delta_5(t)] \\ \rightarrow s\mathcal{L}[y] - 2 &= -\mathcal{L}[y] + 4\mathcal{L}[\delta_5(t)] \\ \rightarrow (s+1)\mathcal{L}[y] &= 2 + 4e^{-5s} \\ \rightarrow \mathcal{L}[y] &= \frac{2}{s+1} + \frac{4e^{-5s}}{s+1} \\ \rightarrow y(t) &= 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + 4\mathcal{L}^{-1}\left[\frac{e^{-5s}}{s+1}\right] = 2e^{-t} + 4u_5(t)e^{-t+5} \end{aligned}$$

Here is a graph of the solution; it's clearly discontinuous. So that's another nice feature of  $\delta_a$  and Laplace transforms: we're able to study differential equations even when there are discontinuities.



As a reminder,

$$\mathcal{L}[e^{-\alpha t}] = \int_0^\infty e^{-\alpha t} e^{-st} dt = \int_0^\infty e^{-(s+\alpha)t} dt = -\left. \frac{e^{-(s+\alpha)t}}{s+\alpha} \right|_0^\infty = \frac{1}{s+\alpha}$$

so long as  $s > -\alpha$ . Also,

$$\mathcal{L}[u_a(t)e^{-t+b}] = e^b \int_a^\infty e^{-(s+1)t} dt = \frac{e^{-as-a+b}}{s+1}.$$

So in our example,  $a = b = 5$ .

## 5 Section 6.5: Convolutions

Suppose we know that  $\mathcal{L}^{-1}[F] = f$ ,  $\mathcal{L}^{-1}[G] = g$  and we want to find  $\mathcal{L}^{-1}[FG]$  in terms of  $f$  and  $g$ . Here, by  $FG$ , I mean the function which is just the pointwise product of  $F$  and  $G$ . So  $(FG)(s) = F(s)G(s)$ . Now, by definition,  $F(s) = \int_0^\infty f(\tau)e^{-s\tau} d\tau$  and  $G(s) = \int_0^\infty g(u)e^{-su} du$ . I'm using different variable names just so we're clear that it's two Laplace transforms here. Then the product is a product of two integrals but we can move one into the other since the integration is in different variables:

$$F(s)G(s) = \int_0^\infty \left( \int_0^\infty f(\tau)e^{-s\tau} d\tau \right) g(u)e^{-su} du = \int_0^\infty \int_0^\infty f(\tau)g(u)e^{-s(\tau+u)} d\tau du.$$

The last equality here comes from the fact that since the order doesn't matter, we're thinking of this as a double integral. Let's do a coordinate change:  $t = \tau + u$ ,  $dt = d\tau$ . Then the  $\tau = 0$  bound is changed to  $t = u$ . In the  $tu$ -plane, the domain of integration is in the 1st quadrant and is the triangular region **below** the line  $u = t$ . Then the last integral above becomes

$$\int_0^\infty \int_u^\infty f(t-u)g(u)e^{-st} dt du = \int_0^\infty \int_0^t f(t-u)g(u)e^{-st} du dt = \int_0^\infty \left( \int_0^t f(t-u)g(u) du \right) e^{-st} dt.$$

The first equality here is obtained by switching the integration order. This requires us to then alter the bounds as well which is easy enough since we know the integration region. The integration now looks like a Laplace transform of something! Namely, if we define  $f * g$  by  $(f * g)(t) = \int_0^t f(t-u)g(u) du$ , then we see that  $F(s)G(s) = \mathcal{L}[f * g]$ . In conclusion:

**Proposition 5.1.**  $\mathcal{L}[f * g] = \mathcal{L}[f] \cdot \mathcal{L}[g]$  and  $\mathcal{L}^{-1}[FG] = \mathcal{L}^{-1}[F] * \mathcal{L}^{-1}[G]$ .

One way to interpret this is that the Laplace transform takes a “fancy” product of two functions given by  $*$  and gives you something which is equal to the usual multiplication of the individual Laplace transforms. Defining this  $*$  operator can be motivated in this way but it's actually important in many other areas of math as well, beginning with discrete variables, for example, and also important to probability theory and statistics and Fourier analysis.<sup>1</sup> Hence, let's define it officially.

**Definition 5.2.** The **convolution** of two functions  $f$  and  $g$  defined on the nonnegatives is

$$(f * g)(t) = \int_0^t f(t-u)g(u) du.$$

Note that because the inverse Laplace transform is unique and regular multiplication is commutative, so is  $*$ :  $f * g = g * f$ . Moreover,  $f * g$  is obtained by integration so it means that  $f * g$  is more differentiable than the derivatives afforded to us by  $f$  and  $g$ .

So now, how do we use it for solving ODEs? Since  $f * g$  requires integration, it might be difficult in general to compute but since we know some of its properties, we can at least make some nice conclusions about solutions to ODEs even if we can't explicitly write something down. But real life is like that; the function describing the stock market surely exists but no one knows how to write it down.

**Example 5.3.** Momentarily, we'll consider any function  $f(t)$  as a forcing term in a harmonic oscillator model. To get started, first consider the equation  $y'' + py' + qy = \delta_0(t)$  with  $y(0) = y'(0) = 0$ . Let  $\zeta(t)$  be a solution to this simpler IVP. Note that it has the following property:

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<sup>1</sup>“Intuitive Guide to Convolution”; <https://betterexplained.com/articles/intuitive-convolution/>

$$\mathcal{L}[\zeta''] + p\mathcal{L}[\zeta'] + q\mathcal{L}[\zeta] = \mathcal{L}[\delta_0] = 1 \rightarrow \mathcal{L}[\zeta] = \frac{1}{s^2 + ps + q}.$$

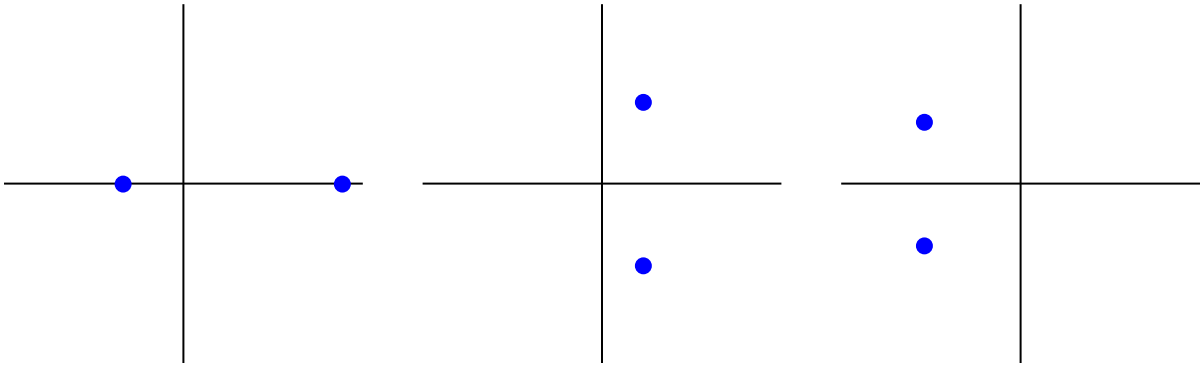
We could do PFD and decompose this, depending on whether the quadratic is irreducible (which depends on  $p, q$ ). We could then find an explicit description of  $\zeta$ . But what we're interested in is if we have a more interesting forcing term  $f(t)$ . Applying  $\mathcal{L}$  to the same equation but now with forcing term  $f(t)$ , we find  $\mathcal{L}[y] = \mathcal{L}[\zeta]\mathcal{L}[f]$ . So then  $y(t) = \mathcal{L}^{-1}[\mathcal{L}[\zeta]\mathcal{L}[f]] = (\zeta * f)(t)$ . So now, no matter how crazy  $f$  is, we can actually write down an analytic solution  $\zeta * f$  for the forced harmonic oscillator with initial conditions  $y(0) = y'(0) = 0$ . If we had other initial conditions, we could tweak  $\zeta$ . Of course, this analytic solution is not in closed form; i.e. presented without an integral. It may be difficult to give a closed-form description but it's useful even presented in the integration way since computer software can give good numerical approximations for the integrals of functions.

## 5.1 Poles of $\mathcal{L}[f]$

One thing we haven't really touched on but is hopefully apparent: many of the Laplace transforms are functions which diverge at isolated points. For example, take  $y'' + py' + qy = 0, y(0) = 0, y'(0) = 1$ . Then the same  $\zeta$  above satisfies this harmonic oscillator (which can be recast as a linear system). For example, if we take  $p = 5, q = 6$  then

$$\mathcal{L}[\zeta] = \frac{1}{s^2 + 5s + 6}$$

diverges as  $s$  approaches  $-2$  or  $-3$ . So these two points  $s = -2, -3$  are called **poles** of  $\mathcal{L}[\zeta]$ . If we allowed  $s \in \mathbb{C}$  rather than just  $\mathbb{R}$ , then we could also consider, say,  $p = -1, q = 1$  which gives us poles at  $(1 \pm i\sqrt{3})/2$ . Note that since  $s^2 + ps + q$  is the characteristic polynomial for the 2nd order ODEs, then these poles are exactly the eigenvalues. We can plot them in the complex plane; the ones with nontrivial imaginary part come in conjugate pairs.



In the pictures, we have three examples; one with real roots of opposite signs, one with complex roots with positive real part, and one with complex roots with negative real part. We can easily then know the behavior of solutions. For example, the location of poles of the first one indicate that we have a saddle point. For the other two, we have spiral sources and spiral sinks, respectively. Recall that we say an equilibrium point (for 2-dim linear systems) is **stable** if two linearly independent solutions converge to it as  $t \rightarrow +\infty$  and is **unstable** if instead, they converge as  $t \rightarrow -\infty$ . We then see that if the poles are located in the left half of the complex plane, that we have a stable equilibrium point since we would have exponential decay. Or if only one pole is on the left, we know there is one line of solutions that converge to the point as  $t \rightarrow +\infty$ . None of this discussion is really new to us; we already knew this from Ch. 3. But the point is that the Laplace transform also knows this.

## 6 Appendix: Proof of Lerch's Theorem

I will restrict myself to continuous functions, but the generalization to integrable functions (continuous almost everywhere) is straight forward (replace equal with equal almost everywhere).

**Theorem 6.1** (Lerch's Theorem). *If  $f, g : [0, \infty) \rightarrow \mathbb{R}$  are continuous and  $\mathcal{L}[f] = \mathcal{L}[g]$  for all  $s \geq 0$ , then  $f \equiv g$ .*

We first start with a lemma:

**Lemma 6.2.** *If  $g$  is a continuous function on  $[0, 1]$  and  $\int_0^1 g(t)t^n dt = 0$  for  $n \in \mathbb{Z}_{\geq 0}$ , then  $g(t) \equiv 0$  for  $t \in [0, 1]$ .*

*Proof.* Fix  $\epsilon > 0$ . By the Weierstrass approximation theorem we can find a polynomial  $P(\epsilon, t)$  such that  $|P(\epsilon, t) - g(t)|_\infty := \sup_{t \in [0, 1]} |P(\epsilon, t) - g(t)| < \epsilon$ . The hypothesis on  $g$  plus linearity of integration implies  $\int_0^1 g(t)P(\epsilon, t) dt = 0$ . Taking  $\epsilon \rightarrow 0$ , we get  $\int_0^1 g^2(t) dt = 0$  which implies  $g(t) \equiv 0$ .  $\square$

**Lemma 6.3.** *If  $g$  is a continuous function and if  $\mathcal{L}[g](s) = 0$  for all  $s \geq a$ , then  $g(t) \equiv 0$ .*

*Proof.* Fix  $s^* \geq a$ . At  $s = s^* + n + 1$ , the condition  $\mathcal{L}[g](s) = 0$  implies that

$$0 = \mathcal{L}(g)(s) = \int_0^\infty g(t)e^{-st} dt = \int_0^1 u^n (u^{s^*} g(-\log(u))) du$$

for all  $n = 0, 1, 2, \dots$  where we have made the substitution  $u = e^{-t}$ ,  $du = -u dt$  in the last equality. By the first lemma it follows that  $u^{s^*} g(-\log(u)) \equiv 0$  which implies  $g(t) \equiv 0$  since  $\log : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a convex diffeomorphism.  $\square$

We are now ready to prove Lerch's theorem:

*Proof.* If  $f, g$  are continuous functions on  $[0, \infty)$  and if  $\mathcal{L}[f] = \mathcal{L}[g]$  for all  $s \geq s^*$ , then  $g \equiv f$ . By the linearity of the Laplace transform we have

$$0 = \mathcal{L}[f] - \mathcal{L}[g] = \int_0^\infty (f(t) - g(t))e^{-st} dt$$

for all  $s \geq s^*$ . By the second lemma above it follows that  $f(t) - g(t) \equiv 0$ .  $\square$