π is irrational

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This is a short proof by Ivan Niven.

Proof. Let us suppose that $\pi = a/b \in \mathbb{Q}$ where $a, b \in \mathbb{Z}^+$. We define

$$f(x) = \frac{x^n(a-bx)^n}{n!}$$

for an n to be specified later. We also let $F(x) = f(x) - f''(x) + f^{(4)}(x) - ... + (-1)^n f^{(2n)}(x)$. We observe a few facts.

1. $n!f(x) \in \mathbb{Z}[x]$ and all the terms have degree at least n.
2. $f(a/b-x) = f(x)$. Let $y = a/2b+x$. Then $f(a/b-y) = f(a/2b-x) = f(y) = f(a/2b+x)$. Thus, the function is symmetric across the line $x = a/2b$ and $f(0) = f(a/b) = f(\pi)$.

Since $f(x)$ is a polynomial, we can consider its Taylor series which terminates. Since the term with the smallest degree is x^n, the series starts at n:

$$f(x) = \sum_{k=0} \frac{f^{(n+k)}(0)}{(n+k)!} x^{n+k}.$$

On the other hand, the term with degree x^{n+k} and the LHS and RHS are:

$$\frac{f^{n+k}(0)}{(n+k)!} x^{n+k} = \frac{\binom{n}{k}}{n!} a^{n-k}(-b)^k x^{n+k}.$$

Thus, all $f^{(i)}(0) = 0$ for all $i < n$ and are integers for $i \geq n$. Because of the symmetry of the function f, we have that $f^{(i)}(\pi) \in \mathbb{Z}$ for all i and equals, up to a sign, $f^{(i)}(0)$.

Now, by calculus,

$$\frac{d}{dx}(F'(x)\sin x - F(x)\cos x) = F''(x)\sin x + F(x)\sin x = f(x)\sin x.$$

Then,

$$\int_0^\pi f(x)\sin x\, dx = F'(x)\sin x - F(x)\cos x|_0^\pi = F(\pi) + F(0) \in \mathbb{Z}$$

It must be a positive integral as, on $0 < x < \pi$, $a - bx > 0$ and so $f(x)$ is positive, as is $\sin x$. We can also take the derivative of f to see that its maximum is at $\pi/2$ which is the same as the max of $\sin x$.

$$f(\pi/2) = \frac{a^n \pi^n}{2^{2n}} n! < \frac{a^n \pi^n}{n!}.$$

The integral is at most $a^n \pi^{n+1}/n!$. But for large n, this value becomes arbitrarily small. Once it becomes smaller than 1, we get our contradiction as the integral is supposed to equal a positive integer $F(\pi) + F(0)$. Hence, $\pi \notin \mathbb{Q}$.

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