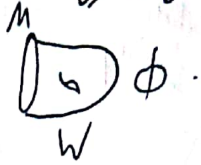
Hirzebruch Signature Thm

Let $\Omega_n^{SO} = \{\text{closed oriented } n\text{-mfds}\}/\sim$. $M \sim N$ if $\exists\, W^{n+1}$, an oriented mfd w/ $\partial W = M \sqcup \bar{N}$.

cobordism

E.g. ~~Ω^{SO}~~,

n	Ω_n^{SO}
0	$\mathbb{Z}$
1	0
2	0
3	0
4	$\mathbb{Z}$ ← generated by $\mathbb{C}P^2$
5	$\mathbb{Z}/2$ ← generated by $SU(3)/SO(3)$ - Wu mfd

We allow  ϕ. This is a group under disjoint union w/ identity = [n-mfds that bound] (or connected sum)

Then, let $\Omega^{SO} = \bigoplus_{n \geq 0} \Omega_n^{SO}$; this is a graded ring under Cartesian product which is compatible w/ addition. A pt is the multiplicative unit.

Indeed: If $M \sim M'$ w/ cobordism W, then $M \times N \sim M' \times N$ b/c $\partial(W \times N)$

$= (M \sqcup \bar{M'}) \times N$

$= (M \times N) \sqcup (\bar{M'} \times N)$.

Thm (Thom): $\Omega^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[p_1, p_2, p_3, \ldots]$, $|p_i| = 4i$.

So this means all the Ω_k^{SO} have only torsion elements when $k \not\equiv 0 \pmod 4$. The proof is not so hard but Thom did the much harder thing & determined Ω^{SO}.

Consider mfds of dim $4n$; then PD let's us study $H_{2n}/\text{Tor} \otimes H_{2n}/\text{Tor} \longrightarrow \mathbb{Z}$; we get an intersection form. $Q(M)$

Claim: Q is symmetric, bilinear, & unimodular.

Defn: Let Q be diagonalized over $\mathbb{Q}$; b^+ = # of positive diagonal elements, b^- = # of neg diagonal elem. & $\sigma(M) = b^+ - b^-$.

b^+ & b^- do not depend on any choices in linear alg, they are homotopy invariants for M & hence, so is σ.

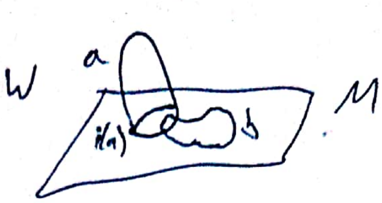
Prop: 1. $\sigma(M \sqcup N) = \sigma(M) + \sigma(N)$ — easy

2. $\sigma(M \times N) = \sigma(M)\sigma(N)$ — Künneth + annoying linear algebra

3. $\sigma(\partial W) = 0$

Pf of (3). We have $\underset{\partial W}{\overset{}{M}} \hookrightarrow W \to (W,M)$ & hence, a LES of pairs; work over $\underline{\mathbb{Q}}$.

$$\begin{tikzcd} H_{2n+1}(W,M) \arrow[r, "i"] \arrow[rr, bend right, "\text{Lefschetz duality}"'] & \underset{\text{PD}}{\overset{H^{2n}}{H_{2n}(M)}} \arrow[r, "j"] & \overset{H^{2n}}{H_{2n}(W)} \end{tikzcd}$$



$\Rightarrow \langle i(a), b\rangle = \langle a, j(b)\rangle$

i.e. i & j are adjoint & hence, have the same rank.

$\operatorname{rk} i = \operatorname{rk} j = \dim H_{2n}(M) - \underbrace{\ker j = \dim H_{2n}(M)}_{\text{exactness}} - \operatorname{rk} i \Rightarrow 2\operatorname{rk} i = \dim H_{2n}(M)$.

Also, if $\begin{matrix} b = i(a) \\ b' = i(a') \end{matrix}$, then $\langle b, b'\rangle = \langle i(a), b'\rangle = \langle a, j(b')\rangle = \langle a, \underbrace{j(i(a'))}_{0}\rangle = 0$.

So $\operatorname{Im} i = \ker j$ is self-annihilating; i.e. the intersection form on $\ker j$ is trivial. $\ker j$ has a dual $(\ker j^*)$ & $H_{2n}(M) = \ker j \oplus (\ker j)^*$; the pieces have the same dim & the pairing is 0 on each summand; hence of the form $\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ (it's symmetric); this is conjugate to $\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$ which has signature 0. ▨

($\oplus$: wrt to PD)

Corollary: $\sigma : \Omega^{SO} \otimes \mathbb{Q} \to \mathbb{Q}$ is a unital ring morphism.

Fact: $\sigma(\mathbb{CP}^{2n}) = +1$; for interest, $S^2 \hookrightarrow \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ is a fibration & $\mathbb{CP}^{2n+1}$ is a boundary

Bigger claim: $\Omega^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \mathbb{CP}^6, \ldots]$

I mean cobordism class in which $\mathbb{CP}^{2n}$ lives

we can compute the Pontryagin classes for $\mathbb{CP}^{2n}$ from Chern classes & they have the right properties.

On to multiplicative sequences & genera

Let $Q(x) \in \mathbb{Q}[[x]]$ be a formal power series of the form $Q(x) = 1 + a_2 x^2 + a_4 x^4 + \cdots$

let x_i be variables of weight 2, $1 \le i \le n$. Then

$$Q(x_1)\,Q(x_2) \cdots Q(x_n) = 1 + \underbrace{a_2 \sum x_i^2}_{\text{weight } 4} + \underbrace{a_4(\text{---})}_{\text{weight } 8} + \cdots$$

We may write this in terms of homogeneous poly $K_r(p_1, \ldots, p_r)$ of weight $4r$ (further view as written as elementary symmetric poly) where the p_j are elementary symmetric poly of x_i^2:

$$Q(x_1) \cdots Q(x_n) = 1 + K_1(p_1) + K_2(p_1, p_2) + \cdots + K_n(p_1, \ldots, p_n) + K_{n+1}(p_1, \ldots, p_n, 0) + \cdots$$

Defn: Let $\{K_r\}_{r=1}^{n}$ be called the <u>multiplicative sequence</u> for Q.

Fact: Suppose x_i, y_i, z_i are some variables satisfying $1 + x_1 + x_2 + x_3 + \cdots = (1 + y_1 + y_2 + \cdots)(1 + z_1 + z_2 + \cdots)$

Then $\displaystyle\sum_{n \ge 0} K_n(x_1, \ldots, x_n) = \left(\sum_{n \ge 0} K_n(y_1, \ldots, y_n)\right)\left(\sum_{n \ge 0} K_n(z_1, \ldots, z_n)\right)$. Hence the ~~name~~ multiplicative seq.

Defn: A <u>genus</u> $\varphi : \Omega^{SO} \otimes \mathbb{Q} \to \mathbb{Q}$ is simply a unital ring morphism.

Given mult seq for $Q \in \mathbb{Q}[[x]]$ of the form from before, define

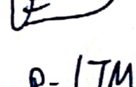
$$\varphi_Q(M^{4n}) = K_n(p_1, \ldots, p_n)[M] \quad \text{where we now view } p_i \in H^{4i}(M, \mathbb{Z}).$$

More generally, $\varphi_Q(M) = K[M]$ when $4 \mid \dim M$; is 0 otherwise.

$$K = \sum_{n \ge 0} K_n(p_1, \ldots, p_n)$$

Lemma: φ_Q is a well-defined genus.

pf: Additivity is clear, multiplicativity follows from the properties of multiplicative seq.

We show $\varphi_Q(\partial W) = 0$. M  W $\quad TW|_M = TM \oplus \underline{\mathbb{R}}$; so

$\partial W = M$

$$p_i(TM) = p_i(TM \oplus \underline{\mathbb{R}}) = p_i(TW|_M). \text{ By Stokes } \int_{\partial W} p_i = \int_W \overset{0}{dp_i}.$$

Clearly, unital

$\boxtimes$

Thm: {genera} $\longleftrightarrow$ {multiplicative seq} $\longleftrightarrow$ {formal power series beginning w/ 1}

Thm (Hirzebruch): Let $L(x) = \frac{x}{\tanh x} = 1 + \frac{x^2}{3} - \frac{x^4}{45} + \frac{2x^3}{945} - \frac{x^4}{4725} + \dots + (-1)^{k-1}\frac{2^{2k} B_k\, x^{2k}}{(2k)!}$. Bernoulli #.

Then $\varphi_L = \sigma$ - signature.

Pf: Being multiplicative, we only need to check $\varphi_L(\mathbb{CP}^{2n}) = \sigma(\mathbb{CP}^{2n}) = +1$;

Let x (deg 2) be the generator of $H^*(\mathbb{CP}^{2n}, \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{2n+1})$. The total Pontryagin class is $p(\mathbb{CP}^{2n}) = (1+x^2)^{2n+1}$; Let $\{K\}$ be the total mult seq for L. Being multiplicative, it satisfies $K((1+x^2)^{2n+1}) = K(1+x^2)^{2n+1} = L(x)^{2n+1}$. So we want to find $\varphi_L(\mathbb{CP}^{2n}) = \left\langle \left(\frac{x}{\tanh x}\right)^{2n+1}, [\mathbb{CP}^{2n}]\right\rangle$.

(*) $K\left(\sum_{i=0}^{\infty} p_i z^i\right) = \sum_{i=0}^{\infty} K_i(p_1,\dots,p_i)\, z^i$

i.e. prove the a_{2n}-coefficient for x^{2n} in $\left(\frac{x}{\tanh x}\right)^{2n+1}$ is 1.

Recall Cauchy Integral formula says: $a_{2n} = \frac{1}{2\pi i}\oint_\gamma \frac{1}{x^{2n+1}}\cdot\frac{x^{2n+1}}{(\tanh x)^{2n+1}}\,dx$, γ around 0.

$= \frac{1}{2\pi i}\oint_\gamma \frac{dx}{(\tanh x)^{2n+1}}$, let $u = \tanh x$, $du = (1-\underbrace{\tanh^2 x}_{u^2})\,dx$

$= \frac{1}{2\pi i}\oint \frac{du}{(1-u^2)\,u^{2n+1}}$; write $\frac{1}{1-u^2} = \sum_{k=0}^{\infty} u^{2k}$; the integral $= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\oint_\gamma \frac{du}{u^{2(n-k)+1}}$.

When $n-k<0$, we have a polynomial which has no pole so the integral vanishes. When $n-k\geq 0$, CIF says $\frac{1}{2\pi i}\oint \frac{du}{u^{2(n-k)+1}} = 2(n-k)^{th}$ derivative of $f \equiv 1$ which is 0 unless $n=k$, in which case, it is the 0^{th} derivative: $f(0) = 1 = a_{2n}$. ∎

Historically, Hirzebruch proved his theorem in one day. In the morning, he learned that Thom showed $\Omega^{SO}\otimes\mathbb{Q} \cong \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \mathbb{CP}^6, \dots]$; so he just needed to find L using $\mathbb{CP}^{2n}$. After computing some of the terms in the ~~Taylor~~ series, he recognized it as the first terms for the Taylor series of $\frac{x}{\tanh x}$.

Suppose we're like Hirzebruch, looking for the formal power series we know $\exists!$ $L(x)$ & $L(x)^{2n+1}$ is such that the coeff for x^{n} is 1. Lets compute some.
(we'll square later)

$n=0$, const term $=1$

$n=1$, we want the ~~linear~~ square term of $L(x)^3$ to have coeff 1. let $b_1 =$ linear coeff for $L(x)$.

$L(x)^3 = (1+b_1x+\dots)^3 = 1+3b_1x+O(x^2)$ so $b_1=\frac{1}{3}$.

$n=2$: want square term to be 1 in $L(x)^5 = (1+\frac{1}{3}x+b_2x^2+\dots)^5 = 1+\frac{5}{3}x+\left(\frac{10}{9}+5b_2\right)x^2+\dots$

So $b_2 = -\frac{1}{45}$.

$n=3$. $L(x)^7 = (1+\frac{1}{3}x-\frac{1}{45}x^2+b_3x^3+\dots)^7 \rightsquigarrow 7b_3+\frac{133}{135}=1$ so $b_3=\frac{2}{945}$.

Let's prove the signature formulas in dim 4 & 8.

Using multiplicativity: $1+K_1^L(p_1)x+K_2^L(p_1,p_2)x^2+\dots = K^L(y_1x)K^L(y_2x) = (1+\frac{1}{3}y_1x-\frac{1}{45}y_1^2x^2+\dots)$
(the x is to keep track of degree) $\cdot(1+\frac{1}{3}y_2x-\frac{1}{45}y_2^2x^2+\dots)$

$RHS = 1+\frac{1}{3}(y_1+y_2)x+(\frac{1}{9}y_1y_2-\frac{1}{45}(y_1^2+y_2^2))x^2+O(x^3)$.

$p_1=y_1+y_2$ (symmetric poly). Comparing deg in x, $K_1^L(p_1)=\frac{1}{3}p_1$. So $\boxed{\sigma(M^4)=\frac{1}{3}p_1(M^4)}$.

Reality check: $\sigma(\mathbb{C}P^2) = 1$ $p_1(\mathbb{C}P^2) = c_2(\mathbb{C}P^2) = \chi(\mathbb{C}P^2)=3$.

$K_2^L(p_1,p_2) = \frac{1}{9}y_1y_2-\frac{1}{45}(y_1^2+y_2^2) = \frac{1}{9}p_2-\frac{1}{45}p_1^2+\frac{2}{45}p_2 = \boxed{\frac{1}{45}(7p_2-p_1^2)} = \sigma(M^8)$

$p_2=y_1y_2$ $\qquad y_1^2+y_2^2 = (y_1+y_2)^2-2y_1y_2$

Eg. $\sigma(K3)=-16 \Rightarrow p_1=-48$. We'll verify $\sigma(K3\times K3)=\sigma(K3)\sigma(K3)=256$.

The total P class $p(X)=(1+x)(1+y)$ where $x=y=p_1(K3)$. For deg reasons, $x^2=y^2=0$.

$1+p_1(X)+p_2(X) = 1+(x+y)+xy$.

$\sigma(X)=\frac{1}{45}(7p_2-p_1^2)$. $p_1^2=(x+y)^2=2xy$ since $x^2=y^2=0$. So $\sigma(X)=\frac{1}{45}(7xy-2xy)$

$=\frac{1}{9}\left(\int_{K3}p_1(K3)\right)^2 = \frac{48^2}{3^2} = 16^2 = 256$.

Remarks: 1. There is a genus for other important invariants as well, like the $\hat{A}$-genus, whose series is $\frac{x/2}{\sinh(x/2)}$. There is a lot of fascinating math about elliptic genera & elliptic cohomology.

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2. Milnor's exotic 7-sphere M is the total space of a principal S^3 over S^4. Clearly, it bounds an 8 mfd. If $M \underset{\text{diff}}{\cong} S^7$, then we can cap off the 8 mfd, call it, N, & compute $\sigma(N)$ which ~~[illegible]~~ should be an integer, computable using $\frac{1}{45}(7p_2 - p_1^2)$. But this turns out to not be an integer, so $M \not\cong S^7$.

A genus φ is determined by $g(x) = \sum_{n \geq 0} \frac{\varphi(\mathbb{CP}^{2n})}{2n+1} x^{2n+1}$, its logarithm.

An elliptic genus φ is one where its logarithm $g(x) = \int_0^x (1 - 2\delta t^2 + \varepsilon t^4)^{-1/2}\, dt$, where $\delta = \varphi(\mathbb{CP}^2)$, $\varepsilon = \varphi(\mathbb{HP}^2)$.

Claim: we get σ when $\delta = \varepsilon = 1$, $\hat{A}$ when $\delta = -1/8$, $\varepsilon = 0$.