Philosophie der Mathematik und Sprache von Kurt Gödel

A revisit of the Incompleteness Theorems and a survey into the foundations of mathematics

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Overview

Goals of the Paper

1. Give the context in which the Incompleteness Theorems emerge
    - Logical Positivism
    - Rudolf Carnaps project on the logical syntax of language
    - Hilberts Formalism

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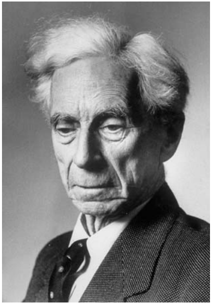
Ignoramus et ignorabimus - Emil du Bois-Reymond (1872)
Wir müssen wissen - wir werden wissen! - David Hilbert (1930)

Ideal Language Philosophy

The IDPs sought to eliminate ambiguity in language; in this way, all statements could be analyzed and be assigned a truth value.

Example (*On Denoting*)

Principle of Indiscernibles: Søren wants to know if J.K. Rowling is the author of the *Harry Potter* series.


Bertrand Russell (1872-1970)

Principia Mathematica

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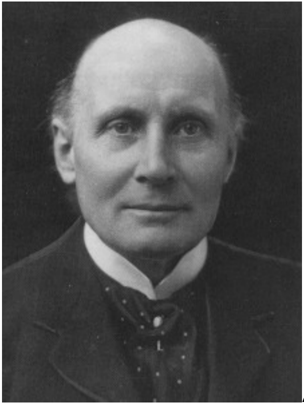
For Russell, mathematics is a template for an ideal language: logical and unambiguous. Thus, he and Whitehead attempted work in the foundations of mathematics: via symbolic logic, describe the axioms and inference rules for which mathematical truths may be proven. Basically, show that classical mathematics is a part of logic.



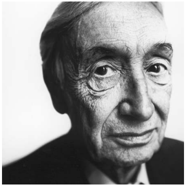
Alfred North Whitehead (1861-1947)

Logical Positivism

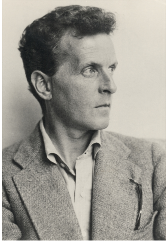
Definition (Verification Principle)

"A sentence has literal meaning if and only if the proposition it expressed was either analytic or empirically verifiable" (A. J. Ayer, *Language, Truth, and Logic*).

For Ayer and many of the Vienna Circle, the role of philosophy is to clarify language and its usage but devoid of any subject matter of its own.



(a) A.J. Ayer (1910-1989)



Ludwig Wittgenstein (1889-1951)

Logical Syntax of Language

Goal of the book as stated by Carnap: "provide a system of concepts, a language, by the help of which the results of logical analysis will be exactly formulable. *Philosophy is to be replaced by the logic of science* - that is to say, by the logical analysis of the concepts and sentences of the science, for *the logic of science is nothing other than the logical syntax of the language of science*" (*Logical Syntax of Language*).

Rudolf Carnap (1891-1970)

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Statements which are true or false solely by their form.

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Statements in mathematics or logic; these make no statements about reality.

Definition (Synthetic)

Statements of the empirical sciences; these have meaning.

Example (Nonexample)

Ethical statements such as "One ought not steal."

Foundations of Mathematics: Platonism

Platonism: an originally Pythagorean belief that there exists nonphysical immutable mathematical entities.

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- C1. Therefore, numbers exist.
- P4. However, numbers, if they exist, must be abstract (non-physical, non-mental) objects.
- C2. Therefore, there exists numbers which are abstract objects.

(John Bigelow and Sam Butchart, "Numbers"). □

Epistemological problem: abstract objects are not perceivable. Which proposition(s) is faulty?

Foundations of Mathematics: Intuitionism

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Foundations of Mathematics: Intuitionism

Conceptualism: there are universal mathematical entities but they are humanmade; hence, conceptual. Intuitionism countenances the use of bound variables to refer to abstract entities only if those entities can be constructed explicitly. Essentially, mathematics has mental but not physical existence. Note: Intuitionists reject the Law of Excluded Middle.


L.E.J. Brouwer (1881-1966)

Foundations of Mathematics: Formalism

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Formalism: mathematics is symbols on a page which we manipulate according to certain rules.
Hilbert thought that mathematics has a meaningful part and a purely formal part. The meaningful part consists of decidable finitary statements such as numbers while the purely formal part consists of ideal statements that involve unbounded quantification over infinite domains such as the natural numbers.


David Hilbert (1862-1943)

Hilbert's Second Problem

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Hilbert hoped for a **consistent** axiomatic description of arithmetic as it is foundational in mathematics. If we have a consistent axiomatic description, then every derivable formula or its negation would have a constructive proof. Thus, there would be no abstract Platonic entities and there would be no need for further construction because we **know** that in principle, we can derive all mathematically true statements. Ideas such as infinity or non-constructible numbers would be purely formal within the language of mathematics and thus, make no ontological statement. Essentially then, Hilbert wanted to **identify truth with provability**.

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- *Habilitationschrift*: Incompleteness Theorems
- Given *privatdozent* position in Vienna and traveled to the US (1933)
- The *Anschluss* (1938), fled to the US (1940)
- Passes away in 1978 at the IAS

The Incompleteness Theorems

The Incompleteness Theorems were originally published in 1930 in the paper, "Über Formal Unentscheidbare Sätze der *Principia Mathematica* und Verwandter Systeme I" ("On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems I").
Here, related systems include the Peano Axioms, Primitive Recursive Arithmetic, and Zermelo-Fraenkel set theory.

Definition

Decidable: a statement P written within the language of a formal system T is said to be decidable if there is a finite algorithm which can, using only the axioms of T, tell us True if P is provable and False if P is not provable.

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Definition

Consistent: a formal system T is said to be consistent if, from the axioms, we cannot derive P and $\neg P$, i.e. we cant derive a logical contradiction.

The First Incompleteness Theorem

Theorem (First Incompleteness Theorem)

For every ω-consistent primitive recursive class κ of formulae there is a primitive recursive class-sign r such that neither $\forall(v, r)$ nor $\neg(\forall(v, r))$ belongs to Conseq(κ) (where v is the free variable of r).

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Theorem (Second Reformulation)

Any consistent formal system T within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of T which can neither be proved nor disproved in T (so there is a statement which is ***undecidable****).*

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For any formal effectively generated system T *including basic arithmetical truths and also certain truths about formal provability, if* T *includes a statement of its own consistency then* T *is inconsistent.*

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For any formal effectively generated system T including basic arithmetical truths and also certain truths about formal provability, if T includes a statement of its own consistency then T is inconsistent.

The first theorem shows that arithmetic is incomplete in the sense that there are arithmetical statements that are formally undecidable; the second shows that if the consistency of the system is expressible within the system itself, then there must be inconsistency somewhere in T.

Proof of the First Theorem

1. We assume T to be sufficiently rich to do arithmetic. Gödel showed that such a system is also rich enough to set up Gödel numbering: assign any statement of T a unique natural number. The assignment is arithmetic in nature.

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4. Now, suppose T has the feature that only true formulas are provable in it (we don't want to prove false statements true!) If G_T were provable in T, then given its content, G_T is false. But only true statements in T are provable and G_T is not true. Therefore, G_T could not be provable and therefore, it is true.

Proof of the First Theorem Cont'd

5. Consider the negation. $\neg G_T$ must be false since G_T is true. $\neg G_T$ says "this sentence (itself) is provable in T." Since it is false, then it is the case that $\neg G_T$ is also unprovable.

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6. Thus, weve shown that G_T and $\neg G_T$ are both unprovable, i.e. G_T is undecidable. Therefore, T is incomplete. □

Note that this proof is constructive (intuitionism).

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    - Let us assume T is inconsistent. That is, we have $P \wedge \neg P$ as true for some P. Then, by addition, form the statement "$P \vee Q$" where Q is just some other proposition. However, we also have $\neg P$ and thus, by disjunctive syllogism, Q.

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    - Then, we could prove anything, including G_T. However, since G_T is true, not every sentence is provable in T. By *modus tollens* on (3), we find that T must be consistent.

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6. $\Leftarrow$ If CON(T) is true then, T is consistent. By the first theorem, so is G_T. Therefore, $G_T \iff$ CON(T). Since G_T is undecidable, so is CON(T). Therefore, T cannot prove its own consistency as CON(T) is undecidable. If it could, then elsewhere in T, there is an inconsistency. □

Second Proof of the Second Theorem

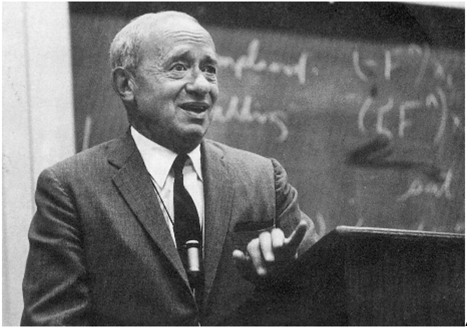
We use a proof by contradiction.

1. Let G_T be the undecidable sentence we constructed earlier. Our RAA hypothesis: *The consistency of the system T can be proven from within T itself.*
2. The first theorem shows that if T is consistent, then G_T is not provable.
3. The proof of the first theorem can be formalized within T, and therefore the statement "G_T is not provable" can be proven in T.
4. But this last statement is equivalent to G_T itself (and this equivalence can be proven in the system), so G_T can be proven in T. We have a contradiction!
5. Therefore, T cannot prove its own consistency. □

Tarski's Indefinability Theorem (1936)

Theorem (Tarskis Indefinability Theorem)

The set of Gödel numbers of the truths of arithmetic is not the extension of any arithmetical formula. In other words, arithmetical truth cannot be defined in arithmetic.



Alfred Tarski (1901-1983)

Tarski's Indefinability Theorem

Implication: Let L_0 be a formal language of arithmetic. L_0 is unable to assert its own truth. Create an extension of L_0, call it L_1, by adding to L_0 the predicate, "is true in L_0". L_1 also has its own system of Gödel numbers. By Tarskis theorem, the Gödel numbers of truths of L_1 are not definable within L_1. We extend yet again and may do so indefinitely but we'll never have a language L which may assert its own truth.

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- The theorems are not saying that we havent found a way of proving consistency and completeness because we havent tried hard enough. They are saying that it will never be the case, no matter how smart or clever we are.
- The theorems are also not making any epistemological claim in the tradition of Skepticism (or any other tradition). In fact, there are no metaphysical claims either.

On Hilbert's Second Problem and Formalism

The Formalist hope of equating truth with provability is defeated by the First Incompleteness Theorem since there will always be true but unprovable sentences in any consistent system; e.g. either G_T or $\neg G_T$ is true but both are unprovable. This applies to all systems strong enough to contain arithmetic which effectively means all of mathematics.

On Hilbert's Second Problem and Formalism

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Secondly, the impossibility of a consistency proof as given by the second theorem along with Tarskis theorem makes giving a finitary proof of the consistency of mathematics impossible.

On *Logical Syntax of Language*

Gödel's interpretation of the philosophical points of Carnap's project:

1. Mathematical intuition, for all scientifically relevant purposes, can be replaced by conventions about the use of symbols. Mathematical intuition of abstract objects is not acknowledged as a source of knowledge by proponents of the syntactical view.

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2. Mathematics, unlike other sciences, does not describe any existing mathematical objects or facts. Rather, mathematical propositions, because they are nothing but consequences of conventions about the use of symbols, are compatible with all possible experience. I.e. they are void of content.
3. The conception of mathematics as a system of linguistic conventions makes the *a priori* validity of mathematics compatible with strict empiricism.

(Richard Tieszen, *After Gödel*)

On *Logical Syntax of Language*

Excerpt from the paper:

"In order for the truths of mathematics to be based solely on linguistic (syntactical) conventions, the syntactical conventions must be consistent. For if they are not consistent, then all statements will follow from them, including all factual (empirical) statements. A rule about the truth of sentences can be called syntactical only if it does not imply the truth or falsehood of any "factual" sentence, that is, one whose truth depends on extralinguistic facts. This requirement follows from the concept of a convention of mathematics upon which its *a priori* nature, in spite of strict empiricism, is supposed to depend."

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In other words, a consistency proof for the syntactical system and inference rules must be given from within the system. By Gödel's second theorem, this is not possible.

Quinean Naturalism/Pragmatism

Quine thinks that we may, in our language and in doing science, posit physical objects as well as abstract concepts such as force, energy, matter, and entities of mathematics because he thinks that scientific theories are our best hope for epistemic inquiry. As long as the systems we use are consistent with experiences and are useful, we go ahead with the ontology ("Two Dogmas of Empiricism").


Willard van Orman Quine (1908-2000)

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Further Reading and Image Credits

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Google Images: Ayer, Brouwer, Carnap, older Gödel, younger Gödel, Hilbert, Quine, Russell, Tarski, Whitehead, Wittgenstein