Symplectic Geometry and Classical Mechanics

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There is a nice article by Henry Cohn about why symplectic geometry is a natural setting for classical mechanics, namely, Hamiltonian mechanics. In the article, he describes why, as we think about classical mechanics, we want a 2-tensor, why we want it to be alternating, nondegenerate, and closed. In short, why do we want a symplectic form? I don't pretend to add much to his thoughts on the matter. But let's go in order; I'll enumerate the questions.

1. Why do we want a 2-tensor?
2. Why do we want it to be nondegenerate?
3. Why do we want it to be alternating?
4. Why do we want it to be closed?

Here is some of my understanding. Let's first consider a concrete setting. If we have, for example, a system with three particles, we can model this on $M = (\mathbb{R}^6)^3 = \mathbb{R}^{18}$ with three position coordinates for each of the particles and similarly for the momentum coordinates. The dynamics of the system; i.e. how things move about, can be described by considering something called the Hamiltonian. This is simply a function $H : M \to \mathbb{R}$ but it's quite powerful. If we know all the forces at play, we can encode them into an H which will represent the total energy of the system.

Okay, so we have this function H but how does it tell us about the dynamics? If a particle is moving about, it should move about as if it were caught in some vector field. Thus, let us try to obtain a vector field X_H from the Hamiltonian H. What should we expect?

- It's clear that if we follow the philosophy that H should determine all the dynamics, then X_H should only depend on dH which tells us about the energy. Moreover, energy should be conserved.
- We also want X_H to depend on dH in a linear fashion. This is because Newton's Laws of Motion, though often formulated in, say, a set of n 2nd-order differential equations, can be written as a set of $2n$ 1st-order differential equations.

Now, we can start answering the questions above.

1. We may ask, "What sort of object takes a 1-form dH and gives us a vector field X_H? The answer is: "A 2-tensor." Well, to be precise, a tensor field which, for us, should be a section of $\mathrm{Hom}(T^*M, TM)$; the dual approach is more convenient: $\mathrm{Hom}(TM, T^*M) \cong T^*M \otimes T^*M$. A 2-tensor field is a linear object which will respect our wish for X_H to depend on dH linearly. Let's call this 2-tensor field ω. Then we may define X_H as follows: $\iota_X \omega = dH$. In other words, $\omega(X, Y) = dH(Y)$ for all vector fields Y. As an observation, note that X_H vanishes at the critical points of H.

2. What other properties would we like ω to have? In terms of physics, we'll like for X_H to always exist uniquely. If this weren't the case, then we either cannot extract any dynamics from the Hamiltonian or we extract different dynamics and get conflicting observations. **Nondegeneracy** of ω guarantees the existence and uniqueness of X_H. This is why we want ω to be nondegenerate.

   To give a more mathematical answer; symplectic geometry has some similarities to Riemannian geometry. If (M, g) is a Riemannian manifold, then since we require g to be nondegenerate, we have a canonical way of identifying TM and T^*M. Similarly, when we have ω as a nondegenerate 2-form on a manifold N, we have a canonical way of identifying TN with T^*N. This seems like a natural feature to consider, especially given the success of Riemannian geometry in physics.

3. Now, we incorporate one of the laws of classical physics: conservation of energy. What does this translate to in our setting? Conservation of energy means that H is constant along the trajectories of X_H which translates to $dH(X) = 0$. In other words, we want $\omega(X, X) = dH(X) = 0$. Thus, when ω is **alternating**, it fits the bill. Hence, we see that the 2-tensor field we want is actually a section of $\bigwedge^2 T^*M$; i.e. ω is a differential 2-form.

   Note that the conservation of energy condition is precisely saying that X_H preserves the level sets of H as X_H is **tangent** to the level sets. Or more precisely, the flow generated by X_H is a family of diffeomorphisms $\varphi_t : M \to M$ which preserve the energy levels (level sets) of H.

   Compare this to the definition of the gradient of a function H; if we choose a compatible almost complex structure J, then we're able to define a Riemannian metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$. Then ∇H is defined as the vector field satisfying $g(\nabla H, Y) = dH(Y)$ for all Y. Thus, ∇H would be **orthogonal** to the level sets of H.

4. In classical mechanics, one of the guiding principles is that we should be able to say how a system is behaving at any given time, whether in the past or future. Thus, here, the natural thing to do if we're interested in what happens at time t is to use a pullback: $\varphi_t^*\omega$. However, ω should be time independent since the laws of classical physics should not depend on time. So we really would like $\varphi_t^*\omega = \omega$. What conditions should ω satisfy to guanrantee this wish? The claim is that ω should be closed: $d\omega = 0$.

   Well, when $t = 0$, $\varphi_t = \mathrm{id}$ so $\varphi_0^*\omega = \omega$. If we can show the derivative with respect to t is zero when $d\omega = 0$, then we will have shown $\varphi_t^*\omega = \omega$. Now,

$$\frac{d}{dt}\varphi_t^*\omega = \lim_{h\to 0} \frac{\varphi_{t+h}^*\omega - \varphi_t^*\omega}{h} = \varphi_t^* \lim_{h\to 0} \frac{\varphi_h^*\omega - \omega}{h} = \varphi_t^* \mathcal{L}_X \omega.$$

   Here, $\mathcal{L}_X\omega$ is the Lie derivative. But Cartan has a nice formula for the Lie derivative of differential forms: $\mathcal{L}_X\omega = d\iota_X\omega + \iota_X d\omega$. By definition, the first term is $d(dH) = 0$. If $d\omega = 0$, then the second term is also zero. Thus, $d\omega = 0$ precisely gives us time-independence.

To summarize briefly the responses to the four questions:

1. We take a **tensor field** because we want to obtain a vector field from dH.

2. **Nondegeneracy** gives us the existence of a unique vector field which tells us all the dynamics of the system.

3. An **alternating** tensor gives us conservation of energy.

4. A **closed** 2-form says the laws of physics are time independent.

Hopefully that convinces you why the definition of a symplectic form naturally arises from physics. But one can ask some further questions.

1. What does symplectic area mean physically?

2. Why should physicists care about symplectic manifolds which are not cotangent bundles?

3. What do symplectomorphisms mean in physics?

I will do my best to give some kind of answer.

1. Let's take one of the simplest examples of a symplectic manifold: $(T^*\mathbb{R}, d\lambda)$; we may think of this as $\mathbb{R}^2$ where $\omega = d\lambda = d(p\,dq) = dp \wedge dq$. Since this is an exact form, if we have some region D with boundary given by a smooth simple closed curve γ, then

$$\int_D \omega = \int_\gamma \lambda$$

   by Stokes' theorem. What does the integral on the right mean? The **action functional** of a Hamiltonian system is typically given as a functional on paths with fixed endpoints of some phase space. It is able to encode all the physical information about the equations of motion for a given classical system. So if we're looking at loops and the Hamiltonian is given as H, then the action is:

$$\mathcal{A}_H(\gamma) = \int_\gamma \lambda - \int_0^1 H(\gamma(t))\,dt.$$

   Well, physicists often take the negative of this. But regardless, the symplectic area of D is the action of γ when $H \equiv 0$. The paths which minimize the action functional are supposed to be the paths that the physical system actually takes. This is the **Principle of Least Action**.

   If we take the unit disk in $\mathbb{R}^2$, the unit circle represents a path which, I believe, corresponds to the motion of a harmonic oscillator. What does the disk itself mean? I suppose we can imagine it as, perhaps, all the possible configurations for an oscillating spring that loses energy overtime (which is compact). So, the path in the phase space will look like some spiral towards the origin.

2. Take, for example, the classical system of the sun, earth, and moon. We can describe their position and momentum in $\mathbb{R}^{18} = T^*\mathbb{R}^9$. However, we're mostly interested in their relative positions to each other so we may quotient by some symmetries, namely that of translations. Thus, we will obtain T^{18}, the 18-torus. The symplectic form on $\mathbb{R}^{18}$ descends since it is translation invariant; however, it is no longer exact. On the other hand, it is still locally exact. So a symplectic manifold, while it might not globally be some phase space, it is locally like some phase space.

3. I don't know how to give a physical interpretation of symplectomorphisms which are not Hamiltonian diffeomorphisms. An easy example of such a symplectomorphism is simply translation on the 2-torus. If we have a Hamiltonian diffeomorphism $\varphi := \varphi_1 : M \to M$, we would like $\varphi^*\omega = \omega$ because the laws of physics should be the same over time. However, if we consider all manners of maps which preserve ω, I don't know what the physical interpretation is.