Some Basics in Spin Geometry

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August 19, 2019

These notes come from Lawson and Michelsohn's *Spin Geometry* and Morgan's *The Seiberg-Witten Equations and Applications to the Topology of Smooth Manifolds.*

1 Motivating Example for Clifford Algebras

Recall that we may realize $SU(2)$ as the group of unit quaternions and thereby identify $SU(2)$ with $S^3 := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ = unit vectors in $\mathbb{H}$. There is a natural group action of S^3 on $\mathbb{H}$ via conjugation:

$$S^3 \times \mathbb{H} \to \mathbb{H}; (\alpha, \lambda) \mapsto \alpha\lambda\alpha^{-1}.$$

This action is clearly trivial on the center of $\mathbb{H}$ which is $\mathbb{R}$. This means then that the action preserves both $\mathbb{R}$ and its complement, the imaginary quaternions: **Im** $\mathbb{H}$. If we study the Lie algebra of $SU(2)$ we'll find that it has three generators over $\mathbb{R}$ which, when multiplied, behave just as $i, j, k \in \mathbb{H}$. Thus, $\mathfrak{su}(2)$ identifies with **Im** $\mathbb{H}$ and the action of S^3 on this Lie algebra is the adjoint action.

If we choose $\alpha \in S^3 - \{\pm 1\}$, then the action preserves the two subspaces: $\mathbb{C}\alpha := \{z\alpha : z \in \mathbb{C}\}$ and $\mathbb{C}\alpha j$, defined similarly. Morgan claims that action of α on this second space is rotation by 2θ where θ is the angle between 1 and α.

2 Clifford Algebras

Let V be a finite dimensional vector space over a field k and let q be a quadratic form on V. Recall that we may define a bilinear form $b(v, w) := \frac{1}{2}(q(v + w) - q(v) - q(w))$. This is called the **polarization** of q. Let

$$T(V) = \bigoplus_{r=0} V^{\otimes r}$$

and $I_q(V)$ be the ideal generated by elements of the form $v \otimes v + q(v) \cdot 1$.
Note: we'll eventually not denote the quadratic form q and also stop using $v \cdot w$ or $v \otimes w$ notation.

Definition 2.1. *The **Clifford Algebra** of (V, q) is the quotient k-algebra $Cl(V, q) := T(V)/I_q(V)$.*

Remark: Recall that all finitely generated k-algebras are isomorphic to some polynomial ring over k, subject to some relations. The quotient by these relations is equivalent to quotient by an ideal.

If we consider the canonical projection $\pi_q : T(V) \to Cl(V, q)$ and V as a subspace of $T(V)$, then we have a natural embedding $\pi_q|_V : V \hookrightarrow Cl(V, q)$. It is not completely trivial to show that this map is injective but it can be done by induction. Note that the quotient gives us the following relations:

1. $v \cdot v = -q(v)1$
2. If k is not of characteristic 2, then $v \cdot w + w \cdot v = -2b(v, w)$ where b is defined as above.

Proposition 2.2. *Let $f : V \to A$ be a linear map into an associative k-algebra with unit, such that $f(v) \cdot f(v) = -q(v)1$ for all $v \in V$. Then f extends uniquely to a k-algebra homomorphism $\tilde{f} : Cl(V, q) \to A$. Furthermore, $Cl(V, q)$ is the unique associative k-algebra with this property.*

Remark: this is a very useful characterization of Clifford algebras. For one, it shows that they are functorial in the following sense. Given a morphism $f : (V, q) \to (W, q')$ which preserves the quadratic forms, i.e. $f^*q' = q$, there is an induced homomorphism $\tilde{f} : Cl(V, q) \to Cl(W, q')$. Given another such morphism $g : (W, q') \to (U, q'')$, we see from the uniqueness that $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$. So we have a covariant functor from the category of k-vector spaces with quadratic forms to the category of Clifford algebras.

One consequence of this fact is that the orthogonal group $O(V, q) := \{f \in GL(V) : f^*q = q\}$ extends canonically to a group of automorphisms of $Cl(V, q)$. Moreover, the embedding sends $O(V, q)$ into the subgroup of inner automorphisms.

Example 2.3. Let $k = \mathbb{R}$, V a real vector space of dimension d, and q be an inner product on V. Let $\{e_1, ..., e_d\}$ be an orthonormal basis of V. Then observe that $e_i^2 = -1$ and $e_i \cdot e_j = -e_j \cdot e_i$ for $i \neq j$. Thus, the dimension of $Cl(V)$ as a real vector space is 2^d.

An important automorphism is the involution $\alpha : Cl(V, q) \to Cl(V, q)$ which extends the map $\alpha(v) = -v$ on V. Since $\alpha^2 = \mathrm{id}$, there is a decomposition of $Cl(V, q)$ into eigenspaces of α:

$$Cl(V, q) = Cl^0(V, q) \oplus Cl^1(V, q).$$

Here, $Cl^i(V, q) := \{\varphi \in Cl(V, q) : \alpha(\varphi) = (-1)^i \varphi\}$. Since $\alpha(\varphi_1 \cdot \varphi_2) = \alpha(\varphi_1) \cdot \alpha(\varphi_2)$, we realize that

$$Cl^i(V, q) \cdot Cl^j(V, q) \subset Cl^{i+j}(V, q)$$

where the indices are taken modulo 2. So we have a $\mathbb{Z}_2$-grading and an algebra which has the above decomposition which satisfies this grading is called a $\mathbb{Z}_2$-graded algebra. $Cl^0(V, q)$ is a *subalgebra* and is called the **even part** of $Cl(V, q)$. $Cl^1(V, q)$ is clearly not a subalgebra but is a subspace. It is called the **odd part**.

Example 2.4. Let $V = \mathbb{R}^n$. Then we have the following:

1. $Cl(\mathbb{R}) = \mathbb{R}[x]/\langle x^2 + 1 \rangle \cong \mathbb{C}$. $Cl^0(\mathbb{R})$ is identified with the reals and $Cl^1(\mathbb{R})$ with the purely imaginaries.
2. $Cl(\mathbb{R}^2)$ is the algebra generated by x, y, subject to the relations $x^2 = y^2 = -1, yx = -xy$. Thus, the algebra identifies with the quaternions $\mathbb{H}$: $x = i, y = j, xy = k$. Note that $\alpha(ij) = \alpha(i)\alpha(j) = (-i)(-j)$. So $k = ij$ is an eigenvector and it generates $Cl^0(\mathbb{R}^2)$. Also, α acting on scalars, i.e. $\mathbb{R} = V^{\otimes 0}$ is trivial. So $\{1, xy\}$ forms a basis for $Cl^0(\mathbb{R}^2)$. Therefore, we may identify $Cl^0(\mathbb{R}^2)$ with $\mathbb{C} \subset \mathbb{H}$.
3. $Cl(\mathbb{R}^3)$ is of dimension 8 and is isomorphic to the polynomial ring $\mathbb{R}[x, y, z]$ modulo the relations $x^2 = y^2 = z^2 = -1, xy = -yx, yz = -zy, xz = -zx$. Then in fact, we have an isomorphism with $\mathbb{H} \oplus \mathbb{H}$. Call the map φ. In the first factor, send $1, i, j, k$ respectively to
$$\frac{1 + xyz}{2}, \frac{xy - z}{2}, \frac{yz - x}{2}, \frac{zx - y}{2}$$
and in the second factor, send $1, i, j, k$ respectively to
$$\frac{1 - xyz}{2}, \frac{xy + z}{2}, \frac{yz + x}{2}, \frac{zx + y}{2}.$$

The subalgebra $Cl_0(\mathbb{R}^3)$ is identified with the diagonal copy of $\mathbb{H}$. When verifying that, say, $\varphi((i,0) \cdot \varphi((j,0)) = \varphi((k,0))$, don't forget the relations. For example, $xy = -yx$.

4. For any inner product space V, we have an isomorphism of algebras $Cl(V) \cong CL_0(V \oplus \mathbb{R})$. Letting e be a unit vector in $\mathbb{R}$, the isomorphism is given by $v_0 + v_1 \mapsto v_0 + v_1 \cdot e$. I think Morgan means it's mapped to $(v_0, 0)+(v_1, 0)\otimes(0, e)$. Then, for example, $Cl_0(\mathbb{R}^4) \cong \mathbb{H}\oplus\mathbb{H}$.

3 Filtration of $Cl(V, q)$

The tensor algebra $T(V)$ has a filtration. Define $\widetilde{\mathcal{F}}^r := \bigoplus_{s\leq r} V^{\otimes s}$. Then $\widetilde{\mathcal{F}}^0 \subset \widetilde{\mathcal{F}}^1 \subset \widetilde{\mathcal{F}}^2 \subset ...$ and $\widetilde{\mathcal{F}}^r \otimes \widetilde{\mathcal{F}}^{r'} \subset \widetilde{\mathcal{F}}^{r+r'}$. Let $\pi_q : T(V) \to Cl(V, q)$ be the quotient map and $\mathcal{F}^i = \pi_q(\widetilde{\mathcal{F}}^i)$. This too is a filtration, terminating in $Cl(V, q)$.

Clearly, multiplication in $Cl(V)$ preserves the filtration in that $\mathcal{F}^i \otimes \mathcal{F}^j \to \mathcal{F}^{i+j}$. Thus, there is an associated graded algebra with the induced multiplication

$$\mathrm{Gr}_{\mathcal{F}^*}(Cl(V)) = \bigoplus_{i=0}^{\infty} \mathcal{F}^i/\mathcal{F}^{i-1}.$$

Then, $\mathrm{Gr}_{\mathcal{F}^*}(Cl(V))$ is naturally isomorphic to the exterior algebra $\Lambda^*(V)$ as vector spaces and may even be thought of as $\Lambda^*(V)$ equipped with a new multiplication.

4 $Pin(V)$ and $Spin(V)$

Let $Cl^\times(V)$ denote the multiplicative group of units of the algebra $Cl(V)$ and definte $Pin(V)$ be the subgroup of $Cl^\times(V)$ generated by elements $v \in V$ with $\|v\| = 1$. Of course, since $v^2 = -1$, then these generating v are units. Let $Spin(V) = Pin(V) \cap Cl_0(V)$. We can similarly define $Spin(V)$ as the kernel of the group morphism $Pin(V) \to \mathbb{Z}_2$ induced by the splitting $Cl_0(V) \oplus Cl_1(V)$.

Observe that if $\{e_1, .., e_n\}$ is an ONB for V, then every products of these e_i are in $Pin(V)$. This means $Pin(V)$ contains a vector space basis for $Cl(V)$ and thus, $Cl(V)$ is the smallest algebra over $\mathbb{R}$ containing $Pin(V)$ as a subgroup of its multiplicative group of units. A similar statement can be made for $Spin(V)$ and $Cl_0(V)$.

Corollary 4.1. *Two real or complex representations of the algebra $Cl_0(V)$ whose restrictions to $Spin(V)$ are isomorphic representations are in fact isomorphic representations of the algebra.*

Also note that the natural action of $O(V)$ on V extends to an action of $O(V)$ on $Cl(V)$ as algebra automorphisms preserving the $\mathbb{Z}_2$ grading. The action is faithful and so induces an embedding of $O(V)$ into $\mathrm{Aut}(Cl(V))$ (algebraic automorphisms). Also $O(V)$ preserves V as a subspace of $Cl(V)$ while $SO(V)$ does so as well and preserves orientation.

$Spin(V)$ acts on $Cl(V)$ by conjugation and this preserves the algebra structure and the $\mathbb{Z}_2$ grading. This is commonly called the adjoint action: $Ad_v(w) = vwv^{-1}$. In the case of Lie groups as we have here, the differential of Ad gives a Lie algebra morphism ad which gives the Lie bracket.

Lemma 4.2. *The conjugation action of $Spin(V)$ on $Cl(V)$ induces a representation of $Spin(V)$ as automorphisms of the Clifford algebra $Cl(V)$. The image of this representation consists of automorphisms which preserve $V \subset Cl(V)$ and the orientation. Thus, we have an induced map $Spin(V) \to SO(V)$. This map is surjective and the kernel is $\{\pm 1\}$. If $\dim V \geq 3$, then the kernel is also the center of $Spin(V)$ and the map presents $Spin(V)$ as the universal covering group of $SO(V)$.*

Proof. Note that this conjugation action of $Spin(V)$ is the restriction of $Pin(V)$ acting by conjugation on $Cl(V)$. We check the representation preserves $V \subset Cl(V)$ and need only check generators, namely unit length elements in V. Let $v, w \in V$ with $\|v\| = 1$. w can be anything.

Note that in general, since $v^2 = -\|v\|^2 1$, then $-\|v+w\|^2 = (v+w)^2 = v^2 + v \cdot w + w \cdot v + w^2$. Thus, $vw + wv = -2\langle v, w\rangle$. Then $vwv = -v^2 w - 2\langle v, w\rangle v$. On the other hand, $v^{-1} = -v/\|v\|^2 = -v$ since $\|v\| = 1$. So $vwv^{-1} = -vwv = v^2 w + 2\langle v, w\rangle v = -(w - 2\langle v, w\rangle v)$.

In general, $\langle v, w\rangle/\|v\|$ is the projection of w onto the line in the direction v. Subtracting off this projection is to make w orthogonal to v. Subtracting off two copies then is reflection across the hyperplane $v^\perp$. We have a minus sign here so then $Ad_v(w) = vwv^{-1} = -R_{v^\perp}$: a reflection followed by multiplication by -1.

Thus, we've shown that $vwv^{-1} \in V$ and is orientation-preserving. $Spin(V)$ then acts on V by even products of reflections in vectors of length 1. It is a classical fact that every element of $SO(V)$ is a product of an even number of reflections. From this, it follows that $Spin(V) \to SO(V)$ is surjective and its kernel is the intersection of $Spin(V)$ with the center of $Cl(V)$.

Next, we show that this intersection is in fact, $\{\pm 1\}$. let $\phi \mapsto \phi^t$ be the antihomomorphism of $Cl(V)$ induced from the map of the tensor algebra which sends $v_1 \otimes ... \otimes v_r \mapsto v_r \otimes ... \otimes v_1$ (reverses the order). This allows us to define a norm $N : Pin(V) \to \mathbb{R}^*$, $\alpha \mapsto \alpha\epsilon(\alpha^t)$ where, if $x = x_0 + x_1 \in Cl(V) = Cl_0(V) \oplus Cl_1(V)$, then $\epsilon(x) = x_0 - x_1$. Since $Pin(V)$ is generated by $v \in V$ with $\|v\| = 1$, then observe that if we have generators v, w, $N(v) = v\epsilon(v) = -v^2 = +1$ and $N(vw) = vw\epsilon(wv) = vwwv = +1$. Thus, N on $Pin(V)$ sends everything to 1. The center of $Cl(V)$ when V is even dimensional is isomorphic to $\mathbb{R}$ and when V is odd dimensional, it is isomorphic to $\mathbb{R} \oplus \mathbb{R}$. When we define N on the center, it simply becomes the squaring map. Thus, the center of $Spin(V)$ is contained in and thus equal to $\{\pm 1\}$.

We now have a natural isomorphism $Spin(V)/\mathbb{Z}_2 \to SO(V)$. We show that when $\dim V \geq 2$, this comes from a **connected** cover $Spin(V) \to SO(V)$ so that $Spin(V)$ is not simply two copies of $SO(V)$ (that would be $O(V)$). It suffices to restrict our attention to a 2 dim subspace $W \subset V$. The preimage of $SO(W) \subset SO(V)$ under this covering map is $Spin(W) \subset Spin(V)$ and the induced map $\pi_1(SO(W)) \to \pi_1(SO(V))$ is surjective. Thus, we just need to prove that $Spin(W) \to SO(W)$ is a non-trivial double cover. Identify $Cl(W)$ with $\mathbb{H}$ and $Spin(W)$ with $S^1 \subset \mathbb{C} \subset \mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$. W is identified with the linear subspace generated by j, k. If we look at the usual conjugate action of S^1 on W, we can look at a direct computation. Let $z = x + iy \in S^1$ and $aj + bk \in W$. Then $(x + iy)(aj + bk)(x - iy) = (x^2 - y^2 + 2xyi)(aj + bk)$. You'll observe that $x^2 - y^2 + 2xyi = (x + iy)^2$. Thus, the conjugation action of $z \in S^1$ on $w \in W$ is simply $z^2 w$. Of course, $SO(W) \cong S^1$ as well but its action is zw; so $Spin(W)$ is a non-trivial double cover of $SO(W)$. We just wrap S^1 twice around itself. □

The dimension of $SO(n)$ can be computed by looking at its Lie algebra $\mathfrak{so}(n)$ whose elements are $n \times n$ skew-symmetric matrices and thus, has dimension $\binom{n}{2}$. This then, is the dimension of $Spin(n)$ as well and the Lie algebras of the two are the same.

Example 4.3. In the case of $Spin(3)$, we may identity this with $S^3 \subset \mathbb{H}$ or $SU(2)$. Conjugation of $Spin(3)$ on $\mathbb{R}^3$ may be viewed as action of S^3 on the imaginary quaternions $\operatorname{Im}\mathbb{H}$. Another view of this action is that it is the usual adjoint action of $SU(2)$ on its Lie algebra which consists of 2×2 skew-Hermitian, traceless matrices (thus, has real dimension 3). The image of this adjoint representation is $SO(3)$.

$Spin(4)$ is the double covering of $SO(4) \cong SU(2) \times SU(2)/\mathbb{Z}_2$. Thus, $Spin(4) \cong SU(2) \times SU(2)$.