# Morse-Bott Theory

Suppose the set of crit pts of $h:(M,g)\longrightarrow\mathbb{R}$ is a submfd $N$ w/ components $N_i$ & on any pt of $N_i$, the Hessian is nondegenerate on orthogonal directions to $N_i$. The # of neg eigenvalues of the Hessian is some $p_i$, the Morse index of $N_i$. We obtain a $p_i$-rank vect bundle over $N_i$ called $\Lambda(N_i)$.

Then $V(\phi) = t^2|\nabla h|^2$ vanishes on the $N_i$ but is large elsewhere. Then, the wavef$^{ns}$ (states) vanish rapidly away from the $N_i$.

Pick an $N_i$, call it $N_0$.

Claim: For large $t$, the low energy spectrum of $H_t$ acting on states localized near $N_0$ converge to the spectrum of $\Delta$ on $N_0$.

Let $M(N_0)$ be a tubular nghd of $N_0$; it can be regarded as the normal bundle.

Let $\tilde{d}$ denote the exterior derivative on $N_0$ which extends to $M(N_0)$.

we find that

$$H_t = \overbrace{\tilde{d}\tilde{d}^* + \tilde{d}^*\tilde{d}}^{=\tilde{\Delta}} + H'$$

acts in transverse directions to $N_0$

For large $t$, $H'$ has a similar form to the $\overline{H}_t$ we saw before (makes sense as the transverse directions are nondegenerate & bear similarities to the Morse setting)

Fix $n \in N_0$; then $H'$ can be restricted to the fiber over $n$, $F_n$.

$H'|_{F_n}$ has a single zero energy state — all other states have energy proportional to $t$.

Rmk: Compare this to the Thom class of a bundle $E^n$ which can be viewed as the unique cohomology class in $H^n_{cv}(E)$ which restricts to the generator of $H^n_c(F)$ on each fiber $F$

Let $|\alpha(m;n)\rangle$ be the zero energy state of $H'$ in the fiber $F_n$ at pt $m$. It is a $p$-form ($p = \text{ind } N_0$).

Claim: Similar to Born-Oppenheimer approx in molecular physics, the deg of freedom transverse to $N_0$ are frozen in their ground state $|\alpha\rangle$ b/c of the large energy associated to any excitation.

$\therefore$ we may write a low energy state $|\psi\rangle$ of $H_t$ in the form $|\psi(n,m)\rangle = |\chi(n)\rangle \otimes |\alpha(m,n)\rangle$. (Künneth Formula or Leray-Hirsch thm)

$$\underset{H^*(M(N_0))}{|\psi(n,m)\rangle} = \underset{\text{“}H^*(N_0)\text{''}}{|\chi(n)\rangle} \otimes \underset{H^*(F_n)}{|\alpha(m,n)\rangle}$$

The caveat is that $|\chi\rangle \in H^*(N_0)$ if $\Lambda(N_0)$ is orientable. If not, then $|\chi\rangle$ is a section of the de Rham complex of $N$ twisted w/ the orientation bundle of $\Lambda(N_0)$.

$|\alpha(m,n)\rangle$ is annihilated by $H'$ } so for large $t$,

the eigenval problem $H_t\psi = \lambda\psi$, $\psi = \chi\otimes\alpha$ reduces

to $\tilde{\Delta}\chi = \lambda\chi$ on $N_0$.

The 0-eigenstates $\chi$ are in 1-1 correspondence to generators of the (twisted) cohomology

The approx we're making is to ignore $\alpha$'s dependence on $N_0$.

This is valid to lowest order in $1/t$.

$\Rightarrow$ nonzero energy states in the approx have nonzero energy in actuality for large $t$

In fact, their energies equal (for large $t$), the nonzero eigenval of the Laplacian on $N$.

We obtain inequalities for Morse-Bott theory.

The contribution of $N_0$ to the Morse polynomial is

$$t^{\rho}\,\bar{P}_t(N_0)$$

ordinary Poincaré poly or twisted Poincaré

$$\hat{P}_t(N_0) = \sum_k b_k(N_0)\,t^k$$

# 3. Killing Vector Fields

Let $(M,g)$ be a cpt Riem mfd

def. A Killing vector field $K$ on $M$ satisfies

$$\mathcal{L}_K g = 0.$$

It may be viewed as an infinitesimal generator of an isometry of $M$ i.e. its flow generates a 1-param family of isometries.

Fact: If $(M,g)$ is cpt, $K$ – Killing vect field, $\eta$ – harmonic form, then $\mathcal{L}_K \eta = 0$

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We fix such a $K$. Let $N = \{$vanishing pts of $K\}$

Regard $K$ as an operator on forms by interior mult.

$$\iota_K$$

Then, let $s \in \mathbb{R}$ be fixed &

$$d_s \doteq d + s\,\iota_K$$

Note that $d_s$ maps a $p$-form to a combination of a $(p+1)$ & $(p-1)$ form.

Let $V_+ = \Lambda^{\text{even}} T^*M$, $V_- = \Lambda^{\text{odd}} T^*M$.

So $d_s : V_\pm \longrightarrow V_\mp$.

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Observe: $d_s^2 = \underbrace{d^2}_{0} + s\,d\,\iota_K + s\,\iota_K d + s^2 \cancelto{0}{\iota_K \iota_K}$

$$= s\,\mathcal{L}_K \quad \text{(Cartan's magic formula)}$$

Also, if $d_s^*$ is the adjoint, using the fact that $K$ is a Killing vect field, we can show that

$$-d_s^* = s\,\mathcal{L}_K$$

Let $H_s = d_s d_s^* + d_s^* d_s$ be our "Hamiltonian."

Main Theorem: The # of zero eigenvalues (multiplicity) of $H_s$ is independent of $s$ (for $s \neq 0$) & indep of any K-invariant Riem metric on $M$.

$$\text{The \# of zero eigenval of } \underset{(s\neq 0)}{H_s} = \sum_k b_k(N)$$

Betti #s.

Moreover, we know that when $s = 0$, $H_s = \Delta$ - Laplacian on $M$.

Then the Hodge thm says:

$$\#\text{ zero eigenval of } \Delta = \sum_k b_k(M).$$

The eigenval of $H_s$ are smooth f^ns of $s$ since the $s$-dependent terms are bounded operators. Then, for very small $s$, the # of 0 eigenval is no bigger than for $s = 0$.

$$\Rightarrow \sum_k b_k(N) \leq \sum_k b_k(M)$$

This is not true in general, of course. $N$ is specifically the fixed pts of flow generated by $K$.

In determining the # of 0-eigenval of $H_s$, for $s \gg 0$, we can express the Hirzebruch signature of $M$ in terms of $N$.

We also obtain a version of the Lefschetz Fixed point thm. where the contribution of each component of $N$ is an integer (its signature).

Lastly, dropping the condition that $K$ is a Killing vect field, we can obtain from the $s \to \infty$ limit of $H_s$ a proof of the Poincaré-Hopf thm.

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These are all variants of the proofs based on the index theorem.

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Let's return to our main goal: Count the zero eigenvalues of $H_s = d_s d_s^* + d_s^* d_s$ $(s \neq 0)$

Note: If $H_s \psi = 0$, then $0 = \langle H_s \psi, \psi \rangle$

$$= \langle d_s d_s^* \psi, \psi \rangle + \langle d_s^* d_s \psi, \psi \rangle$$

$$= \underbrace{|d_s^* \psi|^2}_{\geq 0} + \underbrace{|d_s \psi|^2}_{\geq 0}$$

$\Rightarrow d_s \psi = d_s^* \psi = 0.$

So $H_s \psi = 0$ iff $d_s \psi = d_s^* \psi = 0.$

Hence, if $\psi \in \operatorname{Ker} H_s$, then $\psi \in \operatorname{Ker} d_s^2 = \operatorname{Ker} \mathcal{L}_K$.

So $\psi$ is invariant under the isometries generated by $K$.

So we restrict our attention to $\overline{V} \doteq \operatorname{Ker} \mathcal{L}_K$.

Since $d_s^2 = 0$ in $\overline{V}$, view $d_s$ like a coboundary operator.

Using similar techniques as Hodge theory, one finds the # zero eigenval of $H_s = \dim\left(\ker d_s / \operatorname{Im} d_s\right)$.

The definition of $d_s$ can be made independent of a metric since it relies only on the vect field $K$. So it is indep of $K$-invariant Riem metrics.

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To show the # of 0-eigenval is indep of $s$ (so long $s \neq 0$), we conjugate by $e^{\lambda P}$ (I don't think $P$ is the momentum operator).

This does not change the dim of $\ker d_s / \operatorname{Im} d_s$.

So $e^{-\lambda P} d_s e^{\lambda P} = e^{-\lambda} d_{s'}$, $s' = s e^{2\lambda}$. Tuning $\lambda$, we see have our $s$-independence when $s \neq 0$.

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These arguments work also on counting the # of even or odd 0-energy states. Let these be denoted $n_+$ & $n_-$ for $H_s$ acting on $V_+$ & $V_-$. Then $n_+$ & $n_-$ are indep of $s$ & $n_+ - n_- = \chi(M)$ ~ Euler characteristic

Next Goal: Let $N_+$ = sum of even Betti #s of $N$.

$N_- =$ —″— odd —″—.

Prove $n_\pm \geq N_\pm$.

Claim: we only need to show one of these since

$n_+ - n_- = N_+ - N_- = \chi(M)$. (b/c $n_+ - n_-$ is indep of $s$)

Assuming this formula,

$n_+ \geq N_+ \Rightarrow N_- + (n_+ \underset{\geq}{-} N_+) = n_- \Rightarrow N_- \leq n_-$.

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Depending on whether $n \doteq \dim M$ is odd/even, we focus on showing $n_+ \geq N_+$ or $n_- \geq N_-$.

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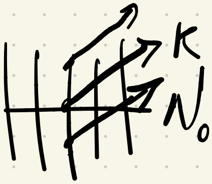
Let $N_0$ be any conn component of $N$ & $\varphi$ a diff form on $N_0$ which is closed but not exact.

Let $M(N_0)$ be a tubular nghd of $N_0$; it has vect bundle structure.

$$\begin{array}{c} M(N_0) \\ \pi \downarrow \\ N_0 \end{array}$$

Let $\hat{\varphi} = \pi^* \varphi$. The action of $K$ on $\hat{\varphi}$ is to lift $K$ to $M(N_0)$ then use interior product:




lift $K$ to fibers to get a vect field on $M(N_0)$.

Then,

$$\begin{tikzcd} H^* M(N_0) \arrow[r, "\iota_K"] & H^{*-1} M(N_0) \\ H^* N_0 \arrow[u, "\pi^*"] \arrow[r, "\iota_K"'] \arrow[ur, phantom, "\circlearrowright"] & H^{*-1} N_0 \arrow[u, "\pi^*"'] \end{tikzcd}$$

$\therefore \ \iota_K \hat{\varphi} = 0$.

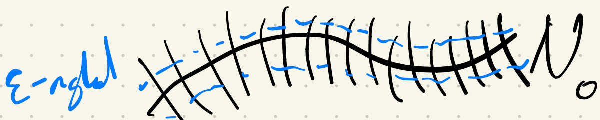
Also, $d\hat{\varphi} = \pi^* d\varphi = 0 \Longrightarrow d_S \hat{\varphi} = 0$.

Also, on $M(N_0)$, it is impossible to satisfy $\hat{\varphi} = d_S \alpha$. This is b/c, on $N_0$, $K \equiv 0$ so $d_S = d$ on $N$. Then $\hat{\varphi} = d_S \alpha \Longrightarrow \varphi = d\alpha$ but $\varphi$ is not exact.

However, on $\partial M(N_0)$, $d\hat{\varphi}$ & $d_S \hat{\varphi}$ are nonzero.

We modify them as follows:

Let $K^2 = g(K, K)$; it vanishes only on $N$. Let $M_\varepsilon$ be pts of $M$ s.t. $K^2 \leq \varepsilon$ for some $\varepsilon > 0$. Let $\varepsilon$ be small so that $M_\varepsilon \subset M(N_0)$.



Let $\phi : M(N_0) \to \mathbb{R}$ be s.t. $\phi|_{N_0} \equiv 1$, $\phi(x) = 0$ for $K^2(x) \geq \epsilon$

Let $\hat{K}$ = $g$-dual of $K$. B/c $K$ is a Killing v.f.

One can show $\iota_K d\hat{K} = -d(K^2)$ (I can't show it)

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Define $\sigma = \phi(K^2) + \frac{1}{s}\phi'(K^2)d\hat{K} + \frac{1}{2s^2}\phi''(K^2)d\hat{K}\wedge d\hat{K}$

$+ \frac{1}{3s^3}\phi'''(K^2)d\hat{K}\wedge d\hat{K}\wedge d\hat{K} + \cdots$

The series terminates since $n = \dim M < \infty$.

Claim: If $n$ even, $d_s\sigma = 0$. If $n$ odd, $d_s\sigma = 0$ except in deg $n$.

Say $n = 2$. Then $\sigma = \phi(K^2) + \frac{1}{s}\phi'(K^2)d\hat{K}$.

$$d\sigma = \phi'(K^2)\underset{-\iota_K d\hat{K}}{\underbrace{d(K^2)}} + \underbrace{\frac{1}{s}\phi''(K^2)\underset{-\iota_K d\hat{K}}{d(K^2)}\wedge d\hat{K}}_{\text{deg 3 so } = 0 \text{ as } n=2} + \overbrace{\frac{1}{s}\phi'(K^2)d^2\hat{K}}^{0}$$

$= -\phi'(K^2)\iota_K d\hat{K}$ ← cancels

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Also, $s\iota_K\sigma = \underset{\substack{\text{b/c deg} = -1}}{s\iota_K\phi(K^2)}_{=0} + \frac{s}{s}\phi'(K^2)\iota_K d\hat{K}$. $\therefore d_s\sigma = 0$.

Based on these patterns, the claim is confirmed.

Let $\chi \doteq \hat{\varphi} \wedge \sigma$. Assume $\varphi$ is even(odd) if $n$ is even (odd).

Then $d_s \chi = (d + s c_k)(\hat{\varphi} \wedge \sigma)$

$= \underbrace{d\hat{\varphi}}_{=0} \wedge \sigma \pm \hat{\varphi} \wedge d\sigma + \overbrace{s c_k \hat{\varphi}}^{=0} \wedge \sigma$

$= \pm \hat{\varphi} \wedge d\sigma$. [I think this is correct. However, Witten says $d_s \chi = 0$.]

Also, $\chi$ is <u>not</u> $d_s$-exact. If it were, that implies $\varphi$ is exact which it is not.

So for every even (or odd) cohom class of $N$, we produced an object $\chi$ which is closed but not exact in the sense of $d_s$.

I think if $[\chi] = [\chi']$, then $[\varphi] = [\varphi']$.

Then, depending on $n$ even/odd, we've shown $n_+ \geq N_+$ or $n_- \geq N_-$.

Now to prove converse inequalities: $N_+ \geq n_+$ & $N_- \geq n_-$.

We compute: $H_s = d_s d_s^* + d_s^* d_s$, let $\hat{K} = {}^{g}\text{dual of } K$

$$= (d + s\iota_K)(d^* + s\underbrace{\hat{K}\wedge}_{\text{wedge operator}}) + (d^* + s\hat{K}\wedge)(d + s\iota_K)$$

$$= dd^* + d^*d + s\,d(\hat{K}\wedge) + s\,\iota_K d^* + s^2 \underbrace{\iota_K \hat{K}\wedge}_{= g(K,K) = K^2}$$

$$+ s\,d^*\iota_K + s\hat{K}\wedge d + s^2 \hat{K}\wedge \iota_K$$

cancels

$$= s(d\hat{K}\wedge - \hat{K}\wedge d)$$

not sure what to do w/ these terms

$$= \Delta + s^2K^2 + s(d\hat{K})\wedge + s(\iota_K d^* + d^*\iota_K) + s^2\hat{K}\wedge\iota_K$$

Witten says we get

$$H_s = \Delta + s^2K^2 + s\big((d\hat{K})\wedge + c(d\hat{K})\big)$$

adjoint of $(d\hat{K})\wedge$.

The potential energy is $V(\phi) = s^2 K^2$ (cf. Morse situation w/ $s^2 |df|^2$)

The proof is similar to the Morse case

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Assume $K$ has only isolated zeros. By Poincaré-Hopf, if the indices add up to nonzero, then $\chi(M) \neq 0 \Rightarrow \dim M = n$ is even.

Claim: When $K$ has only isolated zeros, $N_- = 0$;

$N_+ = \#$ of zeros of $K$.

Pf: Near any zero $A$ of $K$ we can find local coord centered at $A$ for $K$ & $H_s$ can be approx. by a $\overline{H_s}$. Similar to the Morse setting, one can diagonalize $\overline{H_s}$ & $\exists !$ zero eigenval, all others are on the order of $s$.

The one zero eigenval is in $V_+$. So there are $N_+$ states in $V_+$ whose energy does not diverge w/ $s$ & none in $V_-$.

So $n_+ \leq N_+$, $n_- = N_- = 0$. By prev inequality,

$$n_+ = N_+.$$

If $K$ has nonisolated zeros, the discussion is like the

Morse-Bott setting.

Now, $H_s = (d_s + d_s^*)^2$ & $d_s^2 + (d_s^*)^2 = 0$. So

$$H_s = d_s d_s^* + d_s^* d_s. \quad \text{Let } D_s = d_s + d_s^*$$

So $\{\text{zeros of } H_s\} \xleftrightarrow[\text{cor}]{1\text{-}1} \{\text{zeros of } D_s\}$

Decompose $V = V_+ + V_-$ & $D_s = D_{s+} + D_{s-}$.

Note: $D_{s\pm} : V_\pm \longrightarrow V_\mp$.

It can be shown that $\mathrm{Ind}(D_{s+})$ is indep of $s$

& so when $s=0$, $\mathrm{Ind}(D_{s+}) = \chi(M)$.

When $s$ large, $\mathrm{Ind}(D_{s+}) = N_+ - N_- = \chi(N)$.

So $\chi(M) = \chi(N)$, $N$ is the zero set of $K$