The strong Morse Inequalities.

$$\sum M_p t^p - \sum B_p t^p = (1+t)\sum Q_p t^p \qquad Q_p \in \mathbb{Z}_{\geq 0}$$

This eqn is equiv to the assertion that the crit pts model the (co)homology of the mfd M.

It's saying that the difference on the LHS has a "positive" leftover bit. eg. $M_p - B_p = Q_p + \underline{Q_{p-1}}$ some exact things shifted up by a boundary operator ∂.

We already have our (co)boundary operator; it's the d_f we saw from before.

Witten goes on to attempt refining these Morse inequalities. We obtained the inequalities through an approx calculation of the spectrum of H_t. A more accurate calc. could give better bounds.

It's tempting to try computing the higher terms like

$B_p^{(n)}, C_p^{(n)}, \ldots$

However, if $A_p^{(n)}$ vanishes, then these higher terms vanish in $1/t$. I'm not clear on this explanation. He says the higher terms are computable w/ local data; so we don't know if the existence of a crit pt is dictated by global topology or if it is "removable."

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i.e. to gain new info, we study something sensitive to the existence of multiple crit pts. A good candidate is

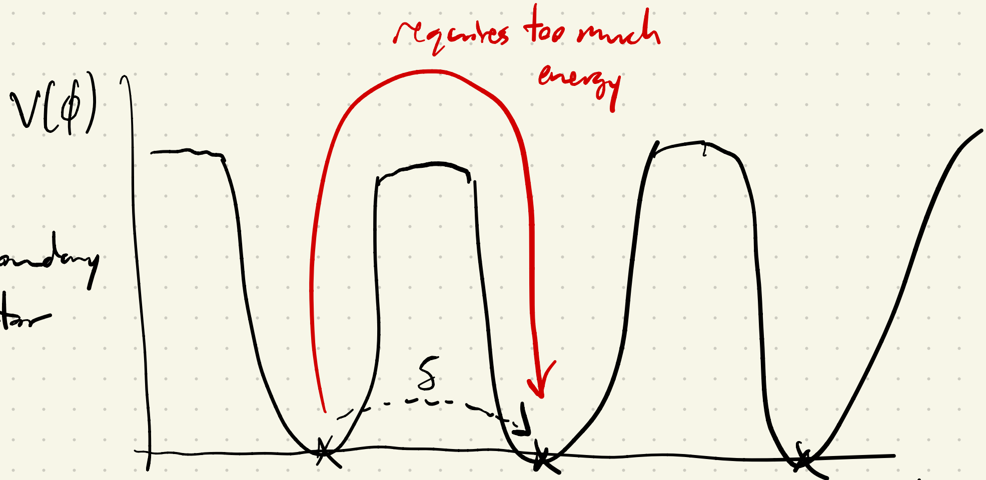
$V(\phi) = t^2 |Dh|^2$ (it has a minimum for each crit pt)

Witten interprets the flow lines of Dh & the boundary operator in terms of tunneling (or instanton corrections)

Rmk. His reference for instanton corrections is Milnor's "Lectures on h-cobordism." Typo?

Notation: X_p = $\mathbb{R}$-vect space generated by index p crit pts

δ = (co)boundary operator



The way written assigns orientation to flow lines is interesting.

At a crit pt A, there is a state $|a\rangle$ of ≈ 0 energy.

Suppose $|a\rangle$ is a p-form. Then let V_A = vect space spanned by negative eigenvectors of $\frac{D^2 h}{D\phi^i D\phi^j}$ at A.

$\dim V_A = p$. Let Γ be a flow line from B (ind $= p+1$) to A.

Let v be the tangent vect of Γ at B & $\tilde{V}_B = \langle v \rangle^{\perp}$ in V_B.

Orientation of $\tilde{V}_B$ is inherited from V_B & flow lines near Γ give mapping $\tilde{V}_B \xrightarrow{f} V_A$. Let $n_\Gamma = \begin{cases} +1, & f \text{ preserves orient} \\ -1, & f \text{ reverses orient} \end{cases}$

Of course, $n(a,b) = \sum_{\Gamma} n_\Gamma$; $\delta|a\rangle = \sum_{b} n(a,b)\cdot|b\rangle$

Instanton calculations show that states <u>not</u> annihilated by $\Delta_\delta \doteq \delta\delta^* + \delta^*\delta$ do <u>not</u> have zero energy. In fact, for large t, the energy is roughly

$$\exp\left(-2t\,|h(A) - h(B)|\right).$$

Let $Y_p = \#\{0\text{-eigenstates of } \Delta_\delta \text{ acting on } X_p\}$

We see that $B_p \leq Y_p$. Does $Y_p = M_p$?

One cannot answer this based on instanton considerations b/c some nonzero energy states may be at approx zero energy & is undetected as nonzero using perturbation theory techniques & instanton calculations. The energy decays more rapidly than $\exp(-2t\,|h(A) - h(B)|)$.

Derivation of $\delta|a\rangle = \sum_b n(a,b)\cdot|b\rangle$

The system described by d_t, d_t^*, H_t can be obtained by canonical quantization of a Lagrangian $\mathcal{L}$ (complicated)

I wonder if this is some equivalence b/w Hamiltonian & Lagrangian formalism, the equiv. furnished by a Legendrian.

$\mathcal{L}$ has terms curvature terms & also seems to have a time coordinate λ.

So we're in a $(\dim M)+1$ spacetime?

I think Witten discards the fermionic terms in $\mathcal{L}$ & assumes the manifold is flat (curvature terms vanish) in order to write a new action:

$$I = \frac{1}{2}\int \Big(g_{ij}\frac{d\phi^i}{d\lambda}\frac{d\phi^j}{d\lambda} + t^2 g^{ij}\frac{\partial h}{\partial\phi^i}\frac{\partial h}{\partial\phi^j}\Big)\, d\lambda$$

metric

The crit pts of $\mathcal{I}$ are the instanton solutions, aka the tunneling paths or flow lines.

Via manipulations:

$$\mathcal{I} = \underbrace{\frac{1}{2}\int \left| \frac{d\phi^i}{d\lambda} \pm t g^{ij} \frac{\partial h}{\partial \phi^j} \right|^2 d\lambda}_{\geq 0} \mp \overbrace{t \int \frac{dh}{d\lambda} d\lambda}^{t\,(h(\infty) - h(-\infty))}$$

$$\Rightarrow \mathcal{I} \geq t\,|h(\infty) - h(-\infty)| \quad \text{[take limits)}$$

& there's equality iff

$$\frac{d\phi^i}{d\lambda} \pm t g^{ij} \frac{\partial h}{\partial \phi^j} = 0$$

Thus, if Γ is a flow line b/w crit pts B & A, then its action is

$$I \doteq \mathcal{I}(\Gamma) = t\,|h(B) - h(A)|.$$

The instanton contributions to H_ϵ are of the order $\exp(-2I)$ which explains why studying instantons cannot answer whether $Y_p = M_p$. (See two pages back)

Rmk. Apparently, when calculating instanton corrections, the next step is usually be the evaluation of the Fredholm determinant for small fluctuations about the classical solution. But the nonzero eigenval 4or bosons & fermions cancel due to SUSY. $\therefore$ we only have zero eigenval of fermions left.

Let Γ be a trajectory b/w A & B. Then

$$\mathrm{Ind}_\Gamma \not{D} = \mathrm{Ind}(A) - \mathrm{Ind}(B)$$

Fredholm index of Dirac operator linearized at Γ

Morse indices

We want to study the cases where the Dirac operator has exactly one 0-eigenvector (aka zero mode or harmonic spinor). In that case $\dim \ker \not{D} = 1$ & possibly,

$$\mathrm{Ind}_\Gamma \not{D} = 1 = \mathrm{Ind}(A) - \mathrm{Ind}(B).$$

In Morse theory, we care about indices differing by 1.

Rmk: Witten gives a physical reason for studying the case where the Dirac operator has exactly one zero mode: it lets us evaluate the action of d_f on very low energy states. & it's relevant that d_f is linear on fermi fields, apparently. As a byproduct, we have the Morse theoretic reasons.

Of course, if the trajectories b/w A & B (index differing by 1) are isolated, then $\not{D}$ has exactly one zero mode & it can be calculated from the classical solution by a SUSY transformation

Witten says: the normalization may be the bosonic zero mode...

"The normalization factor associated with the fermion zero mode cancels in magnitude against the normalization factor associated with the fact that our classical solution is really a 1-parameter family of solutions" (because any solution is still a solution under translation).

I'm not sure what he means. The second part seems to be about quotienting the space of solutions by $\mathbb{R}$ to get a moduli space of trajectories: $\mathcal{M}(A,B)$

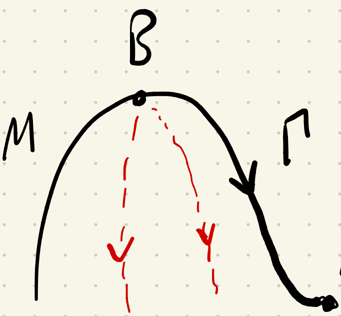
quotienting is the normalization (?)

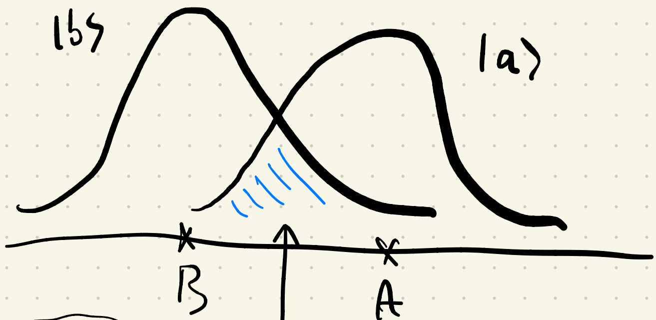
Perhaps he means: $\dim\left[(\ker \not{D}_f)/\mathbb{R}\right] = 0 = \dim \mathcal{M}(A,B)$

Claim: Let $|a\rangle, |b\rangle$ be eigenstates associated to crit pts A & B & Γ is a flow line b/w A & B. Then, the amplitude $\langle b, d_t a\rangle$ of Γ is $\exp(-t\,|h(B)-h(A)|)$

What is amplitude? I don't quite know in physics terms but the eigenstates $|a\rangle$ & $|b\rangle$ concentrate around A & B, resp. So they decay rapidly away from A & B, resp.

However, the decay rate is slowest along trajectories like Γ & the decay rate is $\exp(-t\,h(\phi))$ which looks like $\exp(-t\,|h(B)-h(A)|)$.





Seems Γ is the path of steepest decent so is the fastest path from B to A considering ∇h. However, it is the slowest path of decay for $|b\rangle$ & $|a\rangle$.

overlap is greatest along traj Γ.

Before, we discussed how to give signs to Γ. The physical interpretation seems to use Γ as a propagator of state $|b\rangle$ to $|a\rangle$ & gives the sign of n_Γ based on the sign of the amplitude $\langle b | d_t a \rangle$. This is the WKB approach.

This discussion suggests that the boundary operator is

$$\tilde{\partial}|a\rangle = \sum_b e^{-t(h(B)-h(A))} n(a,b) \cdot |b\rangle.$$

I think the amplitude $\langle b | d_t a \rangle = \sum_\Gamma n_\Gamma e^{-t(h(B)-h(A))}$

However, we can just undo the conjugation by e^{th} in d_t.

The e^{th} don't carry any info in defining $\tilde{\partial}$.

This gives δ which is just a rescaling of $\tilde{\partial}$.

But then $\tilde{\partial}^2 = 0 \iff \delta^2 = 0$. & we know that

$\delta = d_t$ for large t. So $\tilde{\partial}^2 = 0 \iff \delta^2 = 0$
$\iff d_t^2 = 0$

$$\langle b | d_t a \rangle \doteq \int_M \langle b, d_t a \rangle \, d\mathrm{Vol}$$

$$= \int_M b \wedge * d_t a$$

$$= \int_M b \wedge * (da + t \, dh \wedge a)$$

The claim is that

$$\langle b | d_t a \rangle = \sum_\Gamma n_\Gamma e^{-t(h(B) - h(a))} \quad \text{as well}$$

$$= n(a,b) \, e^{-t(h(B) - h(A))}$$

So $\hat{\partial} |a\rangle = \sum_{|b\rangle} \langle b | d_t a \rangle \cdot |b\rangle$

(this is a bit confusing w/ how $B \xrightarrow{\Gamma} A$

So $\hat{\partial}^2 |a\rangle = \sum_{|b\rangle} \sum_{|c\rangle} \left(\langle b | d_t a \rangle - \langle c | d_t b \rangle \right) \cdot |c\rangle$

but $\hat{\partial}$ is a coboundary.