SUSY & Morse Theory

Outline:

1. Brief Intro to Supersymmetry

2. Morse Theory

3. SUSY Quantum Mechanics

4. SUSY Quantum Field Theory

Comment: M. Atiyah was asked to write something to summarize Edward Wittens' work around the time Witten was awarded the Fields Medal in 1990

1. Introduction to SUSY & QFT

In any QFT, there is a Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$

bosons fermions

Bosons: force-carrying particles, integer spin

Fermions: matter particles, half-integer spin, obeys Pauli exclusion principle

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SUSY must have (Hermitian) symmetry operators

Q_i, $i = 1, \ldots, N$ which map $\mathcal{H}^\pm \to \mathcal{H}^\mp$

def: $(-1)^F$ will be the operator $(-1)^F\big|_{\mathcal{H}^\pm} = \pm \mathrm{Id}$

Basic Conditions of SUSY

1. $(-1)^F Q_i + Q_i (-1)^F = 0$. The Q_i are odd

2. If H is the Hamiltonian, then

$$Q_i H - H Q_i = 0$$

Note: H generates time translations (it solves the dynamics)

3. $Q_i^2 = H$

4. For $i \neq j$, $Q_i Q_j + Q_j Q_i = 0$.

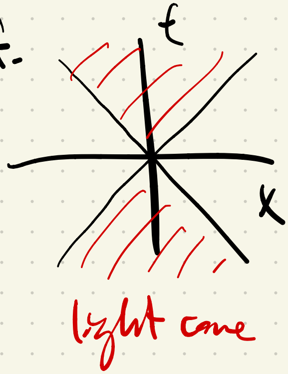
In relativistic settings, we have Lorentz transformations which intermingle space & time translations. In this case, we have further conditions.

Take the simplest case: 1+1 spacetime; has only one momentum operator P.

In this situation, there are only two symmetry operators: Q_1 & Q_2 & they satisfy

5. $Q_1^2 = H+P$, $Q_2^2 = H-P$, $Q_1Q_2 + Q_2Q_1 = 0$

cf.



when $x \pm t = 0$, we're on the boundary of the light cone.

Rmk: symmetric means it commutes w/ the Hamiltonian H.

(5) + Jacobi identity gives

6. $[Q_i, H] = [Q_i, P] = 0$

Observe that (5) also gives: $H = \frac{1}{2}(Q_1^2 + Q_2^2)$
which is positive semi-definite (Q_1, Q_2 are Hermitian)

Since the Q_i are odd, then H & P are even &
$[H, (-1)^F] = [P, (-1)^F] = 0$.

Most Important Question about a SUSY theory

Does there exist a state $|\Omega\rangle$ s.t. $Q_i|\Omega\rangle = 0$ for each i? (*)

If $\exists |\Omega\rangle$, then $H|\Omega\rangle = 0 \Rightarrow$ it has energy $= 0$.

Thus, $|\Omega\rangle$ is a vacuum state; ie. a state of minimum energy.

Rmk: The number of such solutions to (*) is not as important as whether there are any.

Assuming SUSY:

$\exists$ solution to $(*)$ $\Rightarrow$ bosons & fermions have equal mass.

In our experiments, bosons & fermions do not have equal mass. Thus, if SUSY is true in our universe, there is no such solution & vacuum states must have positive energy.

In such a world where $(*)$ has no solutions we say <u>supersymmetry</u> is spontaneously broken.

It is very difficult in general to show if $(*)$ has solutions. It is akin to showing the Dirac operator on a cpt mfd has a zero eigenvalue.

$\therefore$ Indirect methods may be better.

Note: $Q_i|\Omega\rangle = 0 \Rightarrow P|\Omega\rangle = 0$. So restrict attention to $\mathcal{H}_0$ = 0-eigenspace of P. A state in $\mathcal{H}_0$, if annihilated by one Q_i $\Rightarrow$ it is annihil by every Q_i.

Choose one; call it Q.

We frame this now as an index problem:

Since $\mathcal{H}_0 = \mathcal{H}_0^+ \oplus \mathcal{H}_0^-$, decompose $Q = Q_+ + Q_-$.

We have $Q_+ : \mathcal{H}_0^+ \to \mathcal{H}_0^-$ & $Q_- = Q_+^*$ (adjoint)

If $\mathrm{Ind}\, Q_+ \doteq \dim \ker Q_+ - \dim \ker Q_- \neq 0$, then

Q has a zero eigenvalue in $\mathcal{H}_0$.

Claim 1: $\mathrm{Ind}\, Q_+ = \mathrm{Tr}(-1)^F$.

Claim 2: There are SUSY theories w/ $\mathrm{Ind}\, Q_+ \neq 0$ &

∴ there is no spontaneous symmetry breaking.

Further Remarks on SUSY:

The idea is we can interchange bosons & fermions & so they come in pairs. It has never been observed.

But it gives localization: we have some complicated integral over an ∞-dim space of commuting & anti commuting fields.

SUSY says we're integrating something like an exact differential form & so the integral localizes at the critical pts.

This reduces the integral over a fin-dim moduli space.

e.g. instantons, algebraic curves (Donaldson, Gromov-Witten)

2. Morse Theory (Simplest SUSY QM system)

Let (M,g) be a Riem mfld, d, d^* the exterior derivative & its adjoint

Let $Q_1 = d + d^*$, $Q_2 = i(d - d^*)$, $H = \Delta \doteq dd^* + d^*d$

It's easy to check: $Q_1^2 = Q_2^2 = H$, $Q_1 Q_2 + Q_2 Q_1 = 0$.

$\Omega^p = \{p\text{-forms}\}$, p-even = bosons
p-odd = fermions

Let $h: M \to \mathbb{R}$ & $t \in \mathbb{R}$. Define

Note the signs

$$d_t = e^{-ht}\, d\, e^{ht}, \quad d_t^* = e^{ht}\, d^*\, e^{-ht}$$

multiplication by e^{ht} operator

If we let $Q_{1t} = d_t + d_t^*$, $Q_{2t} = i(d_t - d_t^*)$

$$H_t = d_t d_t^* + d_t^* d_t,$$

we have something just as above.

Let α be a diff form, $\{a^k\}$ an ONB of tangent vectors at $p \in M$.

a^k can be viewed as an operator on $\Lambda^* T_p M$ by interior multiplication

$a^k(\varphi) = \iota_{a^k} \varphi$. The dual operator a^{k*} is operation by wedge
(annihilate) (create)

$$d_t \alpha = e^{-th} d(e^{th} \alpha) = e^{-th}(te^{th} dh \wedge \alpha + e^{th} d\alpha)$$

$$= t\, dh \wedge \alpha + d\alpha.$$

$$\therefore d_t = d + t \sum_i \frac{\partial h}{\partial \phi^i} a^{*i} \quad \text{(locally)}$$

Similarly, $d_t^* = d^* + t \sum_i \frac{\partial h}{\partial \phi^i} a^i$ (locally).

This helps us compute

$$H_t = \Delta + \underbrace{t^2 |\nabla h|^2} + t \sum_{i,j} \frac{\partial^2 h}{\partial \phi^i \partial \phi^j} [a^{*i}, a^j].$$

We'll see later that this term is very important &

represents the potential energy.

def. $B_p(t)$ = Betti # w/ d_t: dim of space of d_t-closed p-forms which are not d_t-exact

Claim: $B_p(t) = B_p \doteq B_p(0) \leftarrow$ the usual Betti number

pf: d_t is just d conjugated by e^{ht}, which is an invertible operator. So $\psi \longmapsto e^{-ht}\psi$ is an invertible mapping from closed but not exact p-forms to d_t closed but not d_t exact p-forms. $\boxed{///}$

Moreover, the number of harmonic p-forms in the sense of H_t equals B_p.

Rmk: This independence of t is useful b/c as $t \to \infty$, the spectrum of H_t simplifies. We will then place upper bounds on B_p using crit pts of h.

How does h enter into H_t? Let $\{a^k(p)\}$ be an ONB of T_pM.

Regard it as an operator on $\Lambda^* T_pM$: $\psi \longmapsto \iota_{a^k}\psi$ (interior mult.)

Then a^{k*} is the adjoint: $\psi \longmapsto A^k \wedge \psi$

dual 1-form to a^k.

Rmk: In physics literature,

a^{k*} = fermion creation operator b/c it increases degree w/ wedge

a^k = fermion annihilation operator b/c it decrease degree

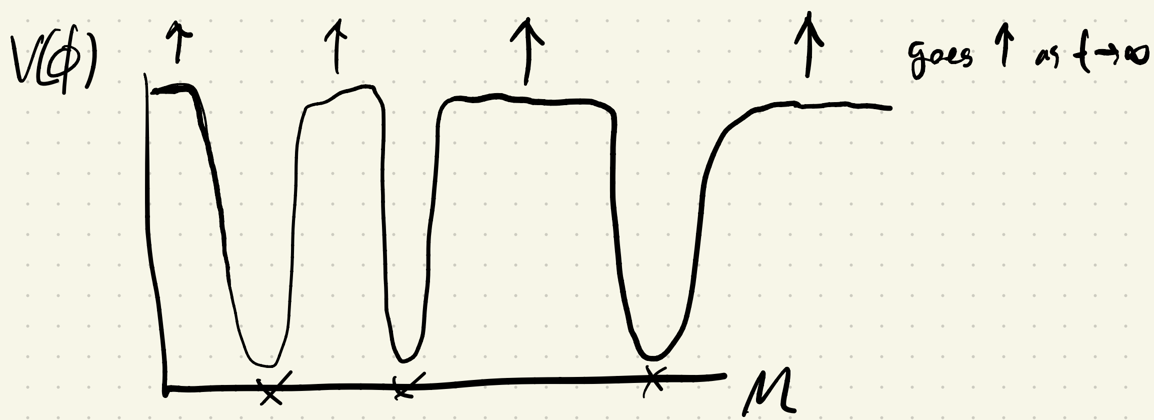
On a Riem mfd, it makes sense to take a covariant 2^{nd} derivative on the dual basis to a^k.

$$H_t = \underbrace{dd^* + d^*d}_{\text{Laplacian}} + \underbrace{t^2|Dh|^2}_{\text{potential}} + \underbrace{t\sum_{i,j}\frac{D^2h}{D\phi^i D\phi^j}[a^{i*}, a^j]}_{\text{other stuff}}$$

$$|Dh|^2 = g^{ij}\left(\frac{\partial h}{\partial \phi^i}\right)\left(\frac{\partial h}{\partial \phi^j}\right)$$

As $t \to \infty$, $V(\phi) \doteq t^2(dh)^2$ becomes large away from the crit pts of h, when $dh = 0$.

Thus, eigenfunctions of H_t are concentrated near the crit pts.



So the eigen f^{ns} approach sums of Dirac-delta f^{ns}.

This alludes to the localization idea from earlier.

Claim: Asymptotic expansion of the eigenvalues in powers of $1/t$ can be calculated w/ local data around crit pts.

Let $h: M \to \mathbb{R}$ be Morse w/ crit pts p^a. The Hessian

$$\frac{D^2 h}{D\phi^i D\phi^j} \text{ is nonsingular}$$

Let M_p = # crit pts w/ Morse index p.

Prop. $M_p \geq B_p$ (Morse Inequalities)

Steps of proof:

1. Using perturbation theory ideas, approximate H_t near crit pts. by an operator $\overline{H_t}$.
2. Compute the spectrum of $\overline{H_p}$ & conclude that to every crit pt of h, there is only one eigenstate of $\overline{H_t}$ whose energy does not diverge w/ t.
3. Not each eigenstate of $\overline{H_t}$ is an eigenstate of H_t but the converse is true.

$\therefore M_p \geq B_p$.

In more details:

Let $\lambda_p^{(n)}(t)$ be the n^{th} smallest eigenval of H_t:

$$\lambda_p^{(n)}(t) = t\left(A_p^{(n)} + \frac{B_p^{(n)}}{t} + \frac{C_p^{(n)}}{t^2} + \cdots\right).$$

Of course, $B_p = \#\{n \in \mathbb{N} : \lambda_p^{(n)}(t) = 0\}$. Also, for large t,

$$\#\{n \in \mathbb{N} : \lambda_p^{(n)}(t) = 0\} \leq \#\{n \in \mathbb{N} : A_p^{(n)} = 0\}$$

It suffices to show: The RHS $= M_p$.

Let ϕ_i be coord s.t. at crit pts p^a, $\phi_i = 0$. Then near p^a,

$$h(\phi_i) = h(0) + \frac{1}{2}\sum \lambda_i \phi_i^2 + O(\phi^3) \text{ for some } \lambda_i.$$

We approx H_t near p^a w/

$$\overline{H}_t = \sum_i \Big(\underbrace{\underbrace{-\frac{\partial^2}{\partial\phi_i^2}}_{\text{Laplacian}} + \underbrace{t^2\lambda_i^2\phi_i^2}_{\text{potential}}}_{\cong\, H_i} + \underbrace{t\lambda_i[a^{i*}, a^i]}_{\cong\, K_i} \Big)$$

other stuff

The correction terms $O(\phi^3)$ can be ignored if we only wish to compute $A_p^{(n)}$ [again relying on the eigenf^{ns} to concentrate at crit pts as $t\to\infty$)

So $\overline{H}_\epsilon = \sum (H_i + \epsilon \lambda_i K_i)$

Claims: ∘ H_i, K_j mutually commute } so can be simultaneously diagonalized.

∘ H_i is the simple harmonic oscillator whose eigenval are well-known: $t|\lambda_i|(1+2N_i)$, $N_i = 0,1,2,\ldots$ These appear w/ multiplicity 1.

∘ Note that the eigenf^{ns} of $\overline{H}_i$ vanish rapidly if $|\lambda_i \phi_i| \gg 1/\sqrt{t}$ $\Rightarrow$ the approx $\overline{H}_\epsilon$ is valid to lowest order in $1/t$.

∘ K_j has eigenval ± 1

Then, the eigenval of $\overline{H}_\epsilon$ are:

(**) $t\sum_i \left(|\lambda_i|(1+2N_i) + \lambda_i \varepsilon_i\right)$, $N_i = 0,1,2,\ldots$, $\varepsilon_i = \pm 1$

Witten says if we restrict $H_t|_{\Omega^p}$, then the # of positive ε_i for the K_i's must be p. Not sure why

For (**) to vanish we need all the $N_i = 0$; $\varepsilon_i = +1$ iff $\lambda_i < 0$.

$\therefore$ Around any crit pt, H_t has exactly one zero eigenfn which is a p-form if the crit pt has Morse index p.

The other eigenval are proportional to t w/ positive coeff.

Then (**) explicitly gives $A_p^{(n)}$ in the spectrum of H_t near p^a.

$\Rightarrow$ For each crit pt, H_t has exactly one eigenstate $|a\rangle$ whose energy does not diverge w/ $t \to \infty$. $|a\rangle \in \Omega^p$ if the assoc. crit pt has index $= p$.

Now, H_t doesn't annihilate all of these $|a\rangle$, just the leading $A_p^{(n)}$ terms. But H_t does not annihilate any other states b/c they have energy proportional to t for large t.

$\therefore \#\{\text{zero energy } p\text{-forms}\} = B_p \leq M_p$. $\blacksquare$

Rmk: This shows that there is a 1-1 correspondence

b/w states $|a\rangle$ s.t. $\overline{H}_t |a\rangle = 0$ } crit pts of h.

Since $\overline{H}_t \approx Q_t^2 = (d_t + d_t^*)^2$, then approx. either

$|a\rangle \in \ker Q_t$ or $Q_t |a\rangle \in \ker Q_t$.

Thus we've found some approx. solutions to:

$$Q_{1t}|a\rangle = Q_{2t}|b\rangle = 0.$$

in this simple case of 2 symmetric operators

This means that the number of SUSY vacua is bounded below by $\sum_p B_p$ ← a topological invariant of M!