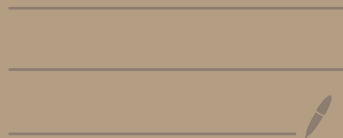


The Poincaré-Hopf Theorem

with Morse theory



Recall: the index of a smooth map $f: M \rightarrow M$

at an isolated fixed pt p is defined locally:

think of $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$; V as a ball around p ;

there are no other fixed pts in \bar{V} (closure). Then the

Gauss map $\nu_{f,p}: \partial V \rightarrow S^{n-1}$, $x \mapsto \frac{x - f(x)}{\|x - f(x)\|}$ ← makes sense bc we're in \mathbb{R}^n locally.

has a degree; so $\text{ind}_p f \equiv \text{deg } \nu_{f,p}$

This is indep of V .

def: Transverse (or nondegenerate) fixed point p .

df_p does not have eigenvalue 1 ; p is a fixed pt of f .

Claim: If p is a transverse fixed pt, then

$$\text{ind}_p f = \text{sgn } \det (\text{Id} - d_p f)$$

Just linearize $\nu_{f,p}$; we're basically following def from there

Now, given a vect field X on M , its zeros are fixed pts of its flow φ^t .

def: let p be an isolated zero of X . Then

$$\text{ind}_p X \stackrel{\circ}{=} \text{ind}_p \varphi^t \text{ for some small } t > 0.$$

IF $f: M \rightarrow \mathbb{R}$ is Morse, then let ∇f be the gradient wrt some Riem metric. $\text{ind}_p \nabla f$ is indep of the metric
{ locally, if φ^t is the flow for ∇f , then $d_p \varphi^t = \exp(t H_p)$

where $H_p = \text{Hess}_p(f)$.

p is nondegenerate $\Rightarrow 0 \notin \text{Spec } H_p \Rightarrow 1 \notin \exp(t H_p)$.

So p is a transverse fixed pt of φ^t .

Diagonalize H_p ; then $\exp(H_p) = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$

where $k \stackrel{\circ}{=} \text{Morse index of } p$. So $(n-k)$ of these are > 1 ;

k of these are < 1 .

Then $\text{Id} - \exp(H_p)$ has $(n-k)$ diagonal entries < 1
 k ~~times~~ > 1 .

$$\text{sgn det}(\text{Id} - \exp(H_p)) = (-1)^{nk}$$

I think we should multiply by $(-1)^n$ b/c we want

$$\underline{\text{ind}_p X = (-1)^k} \quad (*)$$

Poincaré-Hopf | For any vect field X on M , we have
then

$$\sum_{X(p)=0} \text{ind}_p X = \chi(M) - \text{Euler characteristic}$$

pf: The index is defined via the degree of the Gauss map of the flow. Degree is a homotopy invariant & the flow is homotopic to the identity. This shows that the sum of the indices is indep of the vect field X .

So pick a Morse fcn f & let $X = \nabla f$.

By prev computation $(*)$, we have that

$$\sum_{\nabla f(p)=0} \text{ind}_p \nabla f = \sum (-1)^k m_k, \quad m_k = \# \text{ of index } k \text{ crit pts.}$$

Now, if $b_k = \dim \mathcal{B}_k \rightarrow \# \text{ of } M$, then

$b_k \leq m_k$. However, the alternating sum of the ranks of the homology groups is equal to the alternating sum of the ranks of the chain groups from which homology is computed.

$$\begin{aligned} \text{e.g. } b_0 - b_1 &= \dim \ker \partial_0 - (\dim \ker \partial_1 - \dim \text{Im } \partial_1) \\ &= m_0 - \dim \ker \partial_1 \end{aligned}$$

$$b_0 - b_1 + b_2 = m_0 - m_1 + \dim \ker \partial_2$$

\vdots

$$\text{So from } \sum_{\nabla f(p)=0} \text{ind}_p \nabla f = \sum (-1)^k b_k = \chi(M).$$

\square