The Poincaré–Hopf Theorem

with Morse theory

Recall: the index of a smooth map $f: M \to M$ at an isolated fixed pt p is defined locally:

think of $f: \mathbb{R}^n \to \mathbb{R}^n$ & V as a ball around p ; there are no other fixed pts in $\overline{V}$ (closure). Then the Gauss map $\nu_{f,p} : \partial V \to S^{n-1}$, $x \mapsto \dfrac{x - f(x)}{\|x - f(x)\|}$ ← makes sense b/c we're in $\mathbb{R}^n$ locally.

has a degree; so $\mathrm{ind}_p f \doteq \deg \nu_{f,p}$

This is indep of V.

def: Transverse (or nondegenerate) fixed point p.

$d_p f$ does not have eigenvalue 1 & is a fixed pt of f.

Claim: If p is a transverse fixed pt, then

$$\mathrm{ind}_p f = \operatorname{sgn} \det (\mathrm{Id} - d_p f)$$

Just linearize $\nu_{f,p}$ & we're basically following def from there

Now, given a vect field X on M, its zeros are fixed pts of its flow φ^t.

def: Let p be an isolated zero of X. Then

$$\mathrm{ind}_p X \doteq \mathrm{ind}_p \varphi^t \text{ for some small } t>0.$$

If $f: M \to \mathbb{R}$ is Morse, then let ∇f be the gradient wrt some Riem metric. $\mathrm{ind}_p \nabla f$ is indep of the metric

{ locally, if φ^t is the flow for ∇f, then $d_p \varphi^t = \exp(t H_p)$

where $H_p = \mathrm{Hess}_p(f)$.

p is nondegenerate $\Rightarrow 0 \notin \mathrm{Spec}\, H_p \Rightarrow 1 \notin \exp(t H_p)$.

So p is a transverse fixed pt of φ^t.

Diagonalize H_p; then $\exp(H_p) = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$

where $k \doteq$ Morse index of p. So $(n-k)$ of these are >1;

k of these are <1.

Then $\mathrm{Id} - \exp(H_p)$ has $(n-k)$ diagonal entries < 1

k ——〃—— > 1.

$$\operatorname{sgn} \det(\mathrm{Id} - \exp(H_p)) = (-1)^{n-k}$$

I think we should multiply by $(-1)^n$ b/c we want

$\mathrm{ind}_p \varphi^t = (-1)^k$. (*)

Poincaré-Hopf thm | For any vect field X on M, we have

$$\sum_{X(p)=0} \mathrm{ind}_p X = \chi(M) \text{ - Euler characteristic}$$

pf: The index is defined via the degree of the Gauss map of the flow. Degree is a homotopy invariant & the flow is homotopic to the identity. This shows that the sum of the indices is indep of the vect field X.

So pick a Morse fn f & let $X = \nabla f$.

By prev computation (*), we have that

$$\sum_{\nabla f(p)=0} \operatorname{ind}_p \nabla f = \sum (-1)^k m_k \,, \quad m_k = \#\text{ of index } k \text{ crit pts.}$$

Now, if $b_k = k^{th}$ Betti # of M, then

$b_k \leq m_k$. However, the alternating sum of the ranks of the homology gps is equal to the alternating sum of the ranks of the chain gps from which homology is computed.

e.g. $b_0 - b_1 = \dim \ker \partial_0 - (\dim \ker \partial_1 - \dim \operatorname{Im} \partial_0)$

$$= m_0 - \dim \ker \partial_1$$

$$b_0 - b_1 + b_2 = m_0 - m_1 + \dim \ker \partial_2$$

$$\vdots$$

So then $\displaystyle\sum_{\nabla f(p)=0} \operatorname{ind}_p \nabla f = \sum (-1)^k b_k = \chi(M).$ $\blacksquare$