Periodic Orbits of Time-Dependent Hamiltonians

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This is a note concerning periodic orbits of time-dependent Hamiltonians, as discussed in ch. 5 and 6 of *Morse Theory and Floer Homology*. Any "proofs" are really sketches.

1 Hamiltonians and Orbits

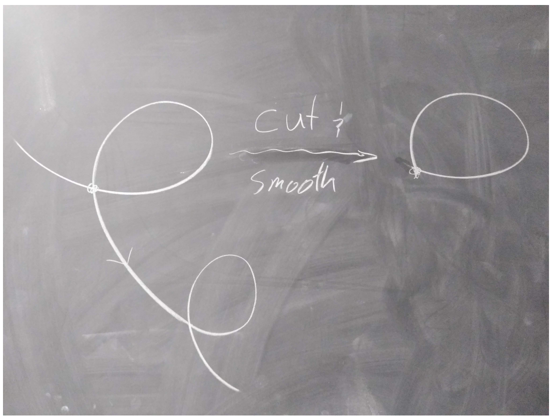
Recall that for a Hamiltonian $H : M \times \mathbb{R} \to \mathbb{R}$ on a symplectic manifold (M, ω), we can define a family of diffeomorphisms φ_t which are almost like flows but lacking the group law.

To do this, we define X_t by requiring $\iota_{X_t}\omega = dH_t$; because of the nondegeneracy of ω, the map $X \mapsto \iota_X \omega$ is an isomorphism between sections of TM and sections of T^*M; i.e. between vector fields and 1-forms. In this case, we're requesting X_t to be such that they map to the exact 1-forms H_t.

Recall that the usual definition of the flow of a vector field X on M is a map $\varphi_t : M \to M$ satisfying $\frac{d}{dt}\varphi_t(p) = X(\varphi_t(p))$. We can adapt this to our situation where X_t change depending on t: $\frac{d}{dt}\psi_t(p) = X_t(\psi_t(p))$. Lastly, a periodic orbit is defined to be a map $x : \mathbb{R} \to M$ satisfies the same ODE: $\dot{x}(t) = X_t(x(t))$.

It's not too hard to show that these periodic orbits x correspond to fixed points of φ_1. Let y be a fixed point of φ_1. Then define $x(t) := \varphi_t(y)$. Observe that $x(0) = \mathrm{id}(y) = y$ and $x(1) = \varphi_1(y) = y$. So $x(0) = x(1)$. Also, $\dot{x}(t) = \dot{\varphi}_t(y) = X_t(\varphi_t(y)) = X_t(x(t))$.

However, observe that in this definition, the periodic orbits might not actually be closed orbits. Depending on the φ_t, we might get a "loop-the-loop" situation like in the image:



Making a Periodic Hamiltonian

In our picture, we see that this periodic orbit may in fact have several loops and thus, have multiple fixed points of φ_1. So the claim isn't that the fixed points always come from periodic orbits, only that a periodic orbit may arise from a fixed point. However, could we modify our situation so that fixed points do come from periodic orbits?

Yes, if we have that the Hamiltonian itself is periodic, i.e. is a function $H : M \times S^1 \to R$, then this is true. How do we get a periodic Hamiltonian K from a non-periodic Hamiltonian H?

2 Periodic Hamiltonians

Suppose we have φ_t coming from a Hamiltonian H_t. We aim to create a **periodic** Hamiltonian K_t from H_t such that φ_1 of H_t and the time 1 "flow" of K_t have the same fixed points. We replace φ_t by $\varphi_{\alpha(t)}$ where $\alpha : [0,1] \to [0,1]$ is smooth. Then

$$\frac{d}{dt}\varphi_{\alpha(t)}(p) = \frac{d\alpha}{dt} X_{H_{\alpha(t)}}(\varphi_{\alpha(t)}(p)) = X_{\alpha'(t)H_{\alpha(t)}}(\varphi_{\alpha(t)}(p)).$$

The map $t \mapsto \varphi_{\alpha(t)}$ is the flow of the Hamiltonian vector field associated with the function $K_t = \alpha'(t)H_{\alpha(t)}$. If we let α be 0 (and flat) near 0 and equal to 1 (and flat) near 1, then $\varphi_{\alpha(1)} = \varphi_1$. So the two have the **same** fixed points. Moreover, K_t can then be extended 1-periodically as it vanishes near the end points of $[0,1]$.

This procedure essentially cuts off the pieces of the orbit which are outside of the time interval $[0,1]$. The fact that α is flat near the end points smooths out our cutting. Though the image shows an image which has a corner, as we approach approach the corner, the derivative becomes zero. So it is in fact, smooth. cf. the cuspidal cubic $\{x^2 = y^3\}$ is smoothly embedded in $\mathbb{R}^2$.

Note that in this modification, if we have a multi-loop-the-loop situation as in the picture, we will lose some of the fixed points when we cut off the "extra" parts. However, Arnold's conjecture gives a lower bound; adding back in those pieces will increase the number of fixed points of φ_1 and so doesn't affect the lower bound.

3 Nondegeneracy

A periodic orbit x of the Hamiltonian is said to be **nondegenerate** if the differential of φ_1 does not have eigenvalue 1. That is, $\det(\mathrm{id} - d_{x(0)}\varphi_1) \neq 0$. A Hamiltonian is nondegenerate if all of its periodic orbits satisfy this property.

The purpose of this condition is to ensure that the fixed points of φ_1 are isolated. Then when M^{2n} is compact, we can consider the diagonal $\Delta \subset M \times M$ and the graph $\Gamma(\varphi_1)$. Both are submanifolds as $\Delta = \Gamma(\mathrm{id})$ and both id and φ_1 are diffeomorphisms. Where the two intersect are the fixed points of φ_1. $d\varphi$ not having eigenvalue 1 at these points precisely means that Δ and $\Gamma(\varphi_1)$ intersect transversally. Then the codim of the intersection is $4n$; i.e. it is a 0-dim manifold. As $M \times M$ is also compact, then there are finitely many fixed points of φ_1.

We ask ourselves, "Does the procedure of turning a Hamiltonian above into a periodic Hamiltonian preserve nondegeneracy?"

Yes, for if we let $\Phi : M \times \mathbb{R} \to \mathbb{R}$ represent the "flows", then precomposing with $(\mathrm{id}, \alpha(t))$, we will obtain a $\Phi_\alpha : M \times \mathbb{R} \to \mathbb{R}$ and our $\varphi_{\alpha(t)}$. So the differential of Φ_α is $d\Phi_\alpha = d\Phi \circ (\mathrm{id}, \alpha'(t))$. Then $d\varphi_{\alpha(t)} = \alpha'(t)d\varphi_t$. Considering $t = 1$, we have $d\varphi_{\alpha(1)} = \alpha'(1)d\varphi_1 = 0$ since α is flat near 1. Then, $\det(\mathrm{id} - d_{x(0)}\varphi_{\alpha(1)}) = 1 \neq 0$. So our procedure preserves nondegeneracy.

Fact: The procedure above is such that H is nondegenerate if and only if K is nondegenerate. This is simply because, if φ_1 is the time 1 flow of H and ψ_1 is the time 1 flow of K, their differentials have the same eigenvalues.

However, if we have a periodic Hamiltonian, note that it isn't necessarily nondegenerate. If we look at the height function on S^2 embedded into $\mathbb{R}^3$ in the standard way, it gives rise to periodic orbits which are actual circles (of latitude) on S^2. If we scale in the right way, we can make it so that φ_1 essentially rotates S^2 on the z-axis by 2π; i.e. $\varphi_1 = \text{id}$. Then every point is a fixed point and this is clearly a degenerate case.

4 Generating New Hamiltonians

Suppose that $H, K : M \times \mathbb{R} \to \mathbb{R}$ are two Hamiltonians with isotopies φ_t and ψ_t respectively. Let $G_t = H_t + K_t \circ \varphi_t^{-1}$. Exercise 7 in Audin and Damien shows that the Hamiltonian vector field of G_t is

$$X_{G_t}(x) = X_{H_t}(x) + (T_{\varphi_t^{-1}(x)}\varphi_t)(X_{K_t} \circ \varphi_t^{-1}(x)).$$

We may deduce from this that $\varphi_t \circ \psi_t$ is the Hamiltonian isotopy generated by G_t. An interesting fact is that even if H and K are both autonomous, the composed Hamiltonian isotopy might not come from an autonomous Hamiltonian.