Oriented Morse Homology: using $\mathbb{Z}$ coefficients

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This short note is based on exercise 1.11 (p. 7) of D. Salamon's *Lectures on Floer Homology.* The main question is: "How shall we define Morse homology with $\mathbb{Z}$ coefficients?" The basic principle is that the compactified moduli spaces $\overline{\mathcal{L}}(x, y)$ (with broken trajectories added in) where the index of x and y differ by 1, are 0-dim manifolds. On a compact manifold M, $\overline{\mathcal{L}}(x, y)$ is finite so if we assigned to each trajectory ± 1, we can sum them. So now the question is, "how shall we assign ± 1?"

1. Choose an orientation on all the unstable manifolds $W^u(x)$ for $x \in \text{Crit}(f)$.
2. Let $z(t)$ be a trajectory connecting x and y where the difference in index is 1; say $\text{Ind}(x) - 1 = k = \text{Ind}(y)$. This means $z'(t) = -\nabla f(z(t))$.
3. Consider the differential of the gradient flow $d\varphi_z^t$. It determines, for large t, a vector space isomorphism
$$T_z W^u(x) \cap \nabla f(z)^\perp \to T_z W^u(y)$$
4. Define $\varepsilon(z) = \pm 1$ depending on whether the isomorphism is orientation preserving or reversing. This works even if the manifold itself is not orientable.

Number 3 needs a bit of explanation. For a fixed t, we look at the tangent space of $W^u(x)$ at $z(t)$. The tangent vector along $z(t)$ is precisely $-\nabla f(z(t))$; then the perpendicular space is an $n-1$ dim hypersurface. It intersects transversally with $W^u(x)$, so the intersection has codim equal to $n - (k+1) + 1$; thus, has dimension k. The dimension of $W^u(y)$ is k so the dimensions for a potential isomorphism work out.

Now, for large enough t, φ^t transports z into a neighborhood of y; I think we want large enough t so that basically $\varphi^t(z) = y$. Also, since φ^t are diffeomorphisms, then their differentials are injective; restricting to $T_z W^u(x) \cap \nabla f(z)^\perp$ doesn't change that, so we have an isomorphism.

A different view is, instead of knocking down a dimension to k, we have a $k+1$ dimensional tangent space above and the gradient gives a single vector direction which can be added to the lower k dim tangent space to give another $k+1$ dim tangent space. We check if the orientation of the one above and below coincide and assign ± 1 in this way.