Morse Theory for Closed 1-Forms

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1 Morse Theory

In Morse theory, if $f : M \to \mathbb{R}$ is a Morse function, then we define a pseudo-gradient field using df and we also may think of the nondegenerate critical points of f through the viewpoint of df being transverse to the zero section of T^*M. It is then a natural question to ask: "Can we extend these notions to **closed** 1-forms?"

Consider the simple but instructive example of the circle. There is a closed but not exact form on S^1 which is often, misleadingly called $d\theta$; it is a volume form. But of course

$$\int_{S^1} d\theta = 2\pi.$$

In general, let V be a closed connected manifold with closed 1-form α. Being closed, we get a well defined homomorphism $\varphi_\alpha : \pi_1(V) \to \mathbb{R}$ defined by $[\gamma] \mapsto \int_\gamma \alpha$. That is because, if η and γ are homotopic loops, say, given by homotopy $F : S^1 \times I$, then

$$0 = \int_{S^1 \times I} F^* d\alpha = \int_{S^1} \gamma^* \alpha - \int_{S^1} \eta^* \alpha.$$

Then, there is a connected covering space $\pi : \widetilde{V} \to V$ associated to $\ker \varphi_\alpha$. That is, the group of deck transformations of $\pi : \widetilde{V} \to V$ is isomorphic to $\pi_1(V)/\ker\varphi_\alpha$ and $\pi_1(\widetilde{V}) \cong \ker\varphi_\alpha$. Moreover, for every $[\tilde{\gamma}] \in \pi_1(\widetilde{V})$, we have that

$$\varphi_{\pi^*\alpha}[\tilde{\gamma}] = \int_{\tilde{\gamma}} \pi^*\alpha = \int_{\pi(\tilde{\gamma})} \alpha = 0.$$

But this precisely means that $\langle \pi^*\alpha, A\rangle = 0$ for each $A \in H_1(V, \mathbb{R})$ (H_1 is the abelianization of π_1 and so if the pairing vanishes on π_1, it certainly vanishes on H_1). Hence, $\pi^*\alpha$ is an **exact** 1-form. We can define a primitive function $\tilde{f}$ for $\pi^*\alpha$ as follows. Choose a base point $y_0 \in \widetilde{V}$ and set

$$\tilde{f}(y) = \int_{y_0}^{y} \pi^*\alpha.$$

This is well-defined since $\pi^*\alpha$ is closed. $d\tilde{f} = \pi^*\alpha$.

This cover $\pi : \widetilde{V} \to V$ is the smallest cover which makes $\pi^*\alpha$ exact; it is called the **integration cover**. If we consider the critical points of $\tilde{f}$, that means we're asking, "For which values y is $(d\tilde{f})_y \equiv 0$?" Well, $(d\tilde{f})_y = (\pi^*\alpha)_y \equiv 0$ if and only if $\alpha_{\pi(y)} \equiv 0$. We'll define what it means to for α to be nondegenerate and also the index of the critical points of α via $\tilde{f}$. We can also define a pseudo-gradient field for α in the same way. If we have a Riemannian metric g on V, then the gradient of a function f is defined by $g(\nabla f, Y) = df(Y)$. Here, we instead define it as $g(\nabla\alpha, Y) = \alpha(Y)$. From there, it makes sense to define a pseudo-gradient field of α.

A pseudo gradient field of α lifts to a pseudo-gradient field of $\tilde{f}$. In the example of S^1, the integration cover is $\exp : \mathbb{R} \to S^1$. Note that if we take the closed but not exact 1-form $d\theta$ (a misleading name), it does **not** have any critical points and so its trajectories just wind around infinitely. Then, lifting $d\theta$, we obtain the exact 1-form dt; its gradient (with respect to the standard metric) should just be ∂_t and the lifted trajectory is simply a path along all of $\mathbb{R}$.

We may also try constructing a Morse theory with the critical points of α. The main difference is that $\widetilde{V}$ will not be compact if α is closed but not exact. What sort of a difference does this lead to? Well, a trajectory between c, d, two critical points of α, lifts to a trajectory between $\pi^{-1}(c)$ and $\pi^{-1}(d)$. There are infinitely many points in these fibers but after choosing a point c_0 in the cover, the path lifting lemma gives us a unique lift of the trajectory. However, we may potentially have infinitely many trajectories to lift with diverging energy because we're dealing with closed 1-forms now. Thus, we should introduce a different coefficient system, namely a Novikov ring. For more details, one can consult Novikov's original papers.

2 Floer Theory

The above serves as a toy example for why Novikov fields appear in Floer theory. Suppose we wish to relax our conditions on a symplectic manifold (M, ω) so that $\pi_2(\omega)$ needn't be 0. We can still do Hamiltonian Floer theory but we'll need to do it on some covering space of the loop space.

Let α_H be the 1-form defined on the space of contractible loops $\mathcal{L}M$ by

$$(\alpha_H)_x(Y) = \int_0^1 \omega(\dot{x}(t) - X_t(x(t)), Y(t))\, dt.$$

If $\pi_2(\omega) = 0$, then α_H is exact and has a primitive which is the usual action functional. But if we don't have this assumption, we will instead consider the space

$$\mathcal{D}M = \{(x, u) : x \in \mathcal{L}M; u : D^2 \to M \text{ is an extension to the disk}\}.$$

We have the equivalence relation: $(x, u) \sim (x, v)$ if

$$\int_{D^2} u^*\omega = \int_{D^2} v^*\omega.$$

Let $\widetilde{\mathcal{L}M} = \mathcal{D}M/\sim$. It has a projection $\pi : \widetilde{\mathcal{L}M} \to \mathcal{L}M$ which comes from just forgetting the disk u. This is a cover of the group $\ker \varphi_\omega$ where $\varphi_\omega : \pi_2(M) \to \mathbb{R}$ is defined by $[f] \mapsto \int_{S^2} f^*\omega$. The expression

$$\mathcal{A}_H(x, u) = -\int_{D^2} u^*\omega + \int_0^1 H_t(x(t))\, dt$$

is an action functional on $\widetilde{\mathcal{L}M}$ and satisfies $\pi^*\alpha_H = d\mathcal{A}_H$. $\pi : \widetilde{\mathcal{L}M} \to \mathcal{L}M$ is the integration cover for $\mathcal{A}_H$.