Note on Milnor Fibrations and Picard-Lefschetz Theory

Sam Auyeung

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These are some notes that mostly come from the brilliant book of John Milnor—*Singular Points of Complex Hypersurfaces*—and also the book *Singularities of Differentiable Maps, v. II* by V. Arnold, S, Gusein-Zade, and A. Varchenko. I mostly record results. For proofs, please consult the original texts.

1 Singularities of Complex Hypersurfaces

Consider a polynomial function $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and let $V_f = f^{-1}(0)$. As a reminder, $\mathbb{C}^{n+1}$ cannot have compact complex submanifolds. Let $K := V_f \cap S_\epsilon^{2n+1}$ where $\epsilon > 0$ is small; this is a **link** of the singularity. Let us consider a Gauss map $g : \mathbb{C}^{n+1} \setminus V_f \to S^1$ defined by $z \mapsto f(z)/|f(z)|$. We will also, on occasion, make the assumption that in some neighborhood of $0 \in \mathbb{C}^{n+1}$, there are no critical points except for possibly, 0. A nonexample is $f(z, w) = z^2w^2$.

Theorem 1.1 (Milnor)**.** *For sufficiently small $\epsilon > 0$, g restricted to $S^{2n+1}(\epsilon) \setminus V_f \to S^1$ is a fibration. Let $F_\theta = g^{-1}(e^{i\theta})$ denote the fibers; F_θ is a non-compact parallelizable smooth manifold of real dimension $2n$.*

Theorem 1.2 (Milnor)**.** *If p is an isolated critical point of f, then each fiber F_θ has the homotopy type of a bouquet of spheres: $F_\theta \simeq \bigvee^\mu S^n$ where μ is strictly positive. The fiber is the interior of a compact manifold with boundary: $\overline{F}_\theta = F_\theta \cup K$, K is as above and is a common boundary for every F_θ. Moreover, K is $(n-2)$-connected; i.e. $\pi_i(K) = 0$ for $i = 0, ..., n-2$.*

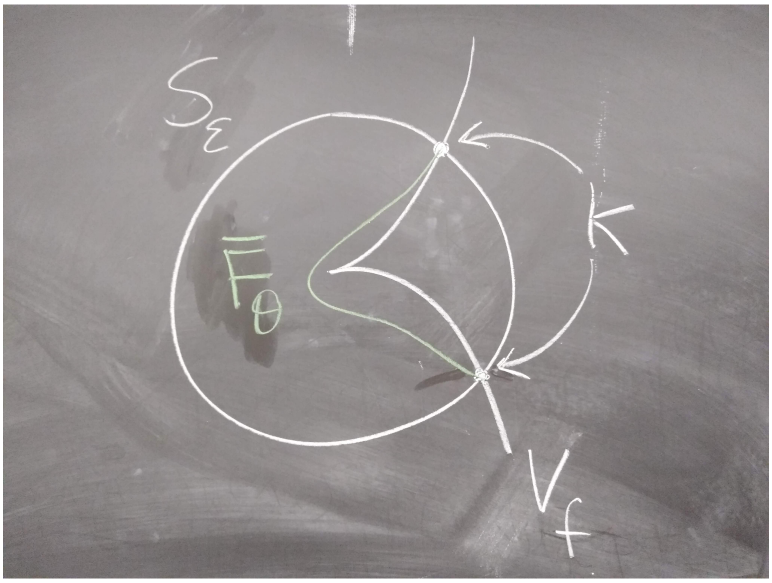
Remark: Note that when we assume the critical point is isolated, we were able to say more. This number μ is called the **Milnor number**. The intuition is that the larger μ is, the more complicated the singularity. The homology of F_θ is only interesting in the middle dimension: $\mu = \dim H_n(F_\theta)$. Also, this is sort of like an open-book decomposition where K is the spine. Or better yet, since all the $\overline{F}_\theta$ have K as the common boundary, this shows that K *links* them together.

We also have a way to compute the Milnor number $\mu(f)$. Simply take the ring of polynomials around 0, $\mathcal{O}_0$ and mod out by the ideal of first partial derivatives of f:

$$\mathcal{A}_f := \frac{\mathcal{O}_0}{\langle \partial_1 f, ..., \partial_{n+1} f \rangle}.$$

This $\mathcal{A}_f$ is an algebra and a vector space. We let $\mu(f) = \dim_\mathbb{C} \mathcal{A}_f$.

Incredibly, the story does not stop there. This compactified manifold with boundary, $\overline{F}_\theta$ is in fact, **diffeomorphic** to $Q := B_\epsilon^{2n+2} \cap V_{f-t}$ where the ball B is bounded by the S_ϵ^{2n+1} from earlier and $V_{f-t} = \{z \in \mathbb{C}^{n+1} : f(z) = t\}$. This t can be taken to be a small nonzero number. One can think of this as perturbing the hypersurface slightly in order to smooth it. In fact, that's how the diffeomorphism is established: find the right vector field to flow $\overline{F}_\theta$ to Q. Both the fiber $\overline{F}_\theta$ and Q are called the **Milnor fiber** (since they are diffeomorphic to each other). See the picture below.



Example 1.3. Let's try our hand at a simple example. Let $f : \mathbb{C}^3 \to \mathbb{C}$ be defined by $f(x, y, z) = x^2 + y^2 + z^2$. As we'll generalize later, we'll see that for small nonzero t, $f^{-1}(t) \cong T^*S^2$. In this case, the singularity arises when we let $t \to 0$. We'll discover that the S^2 is a vanishing cycle in T^*S^2. But away from the zero section, nothing really happens. Thus, we find that the link K is the unit sphere bundle of T^*S^2. So $K \cong SO(3)$ which is in turn diffeomorphic to $\mathbb{R}P^3$. Indeed, it is connected (it should be $(n-2)$ connected where $n = 2$).

What can we say about a fiber F_θ of the fibration $g : S^5_\epsilon \setminus K \to S^1$? Well, we've sort of cheated: we already know that they are diffeomorphic to T^*S^2. But let's just state all the results from above for F_θ anyways. $\dim_{\mathbb{R}} F_\theta = 4$ and $\partial F_\theta = K = \mathbb{R}P^3$. And the homotopy type of F_θ is some wedge of 2-spheres: $\bigvee^\mu S^2$. Here, we see that $\mu = 1$, from the algebraic formula. Lastly, since F_θ is parallelizable, this tells us that $T(T^*S^2)$ is trivial. It is true that $\mathbb{R}P^3$ is also parallelizable (all closed, oriented 3-manifolds are) but in general, we cannot conclude that if a manifold with boundary is parallelizable on the non-boundary part, the boundary is parallelizable.

1.1 Back Tracking

Let's look at some of the arguments for why F_θ has the properties it has. A useful fact for proving F_θ has the homotopy type of a bouquet of spheres is: $\overline{F}_\theta$ is homotopy equivalent to $S_\epsilon \setminus \overline{F}_\theta$. Here is the argument. Take some other angle, say ϕ; then

$$\begin{array}{ccc} F_\phi & \hookrightarrow & S_\epsilon \setminus \overline{F}_\theta \\ & & \downarrow \\ & & S^1 \setminus \{e^{i\theta}\} \end{array}$$

is a fibration. But of course, the base is contractible and so the embedding gives us an isomorphism $\pi_k(F_\phi) \cong \pi_k(S_\epsilon \setminus \overline{F}_\theta)$ for all k. Thus, F_ϕ and $S_\epsilon \setminus \overline{F}_\theta$ are homotopy equivalent. But more over, F_ϕ is diffeomorphic to F_θ and therefore, latter is homotopy equivalent to $S_\epsilon \setminus \overline{F}_\theta$.

Another result is that F_θ has the homology of a point in dimensions less than n and in fact, F_θ is $(n-1)$-connected. To establish the former, we recall what reduced homology and Alexander Duality are.

Take the map $X \to pt$. Then the reduced homology $\widetilde{H}_i(X)$ is defined as the kernel of $H_i(X) \to H_i(pt)$ and the reduced cohomology is $\widetilde{H}^i(X)$ is the cokernel of $H^i(pt) \to H^i(X)$ (induced from the same map but of course, cohomology is contravariant). Alexander Duality states that $\widetilde{H}_i(S\epsilon \setminus \overline{F}_\theta) \cong \widetilde{H}^{2n-i}(\overline{F}_\theta)$. But the latter is zero for $2n - i > n$ as Milnor establishes through some other arguments. Thus, for $i = 0, ..., n-1$, $\widetilde{H}_i(S\epsilon \setminus \overline{F}_\theta) \cong \widetilde{H}_i(F_\theta) = 0$. To see why F_θ is $(n-1)$ connected, see Milnor's book.

1.2 Other Results

We now briefly recalled the definition of the Poincaré-Hopf index of a vector field. Let V be a vector field on an n-manifold M and suppose V vanishes at p. Take a small ball D^n around p and consider the map $u : \partial D^n \to S^n$ which sends $x \mapsto V(x)/|V(x)|$. We can assume that the ball is small enough that we're basically in $\mathbb{R}^n$. The Poincaré-Hopf index is the degree of this map u.

Proposition 1.4 (Milnor)**.** *Let ∇f be the gradient vector field of f (with standard metric on $\mathbb{C}^{n+1}$). The Poincaré-Hopf index of ∇f at 0 is μ.*

Remark: This is another view of μ telling us how complicated the singularity is. The gradient of f contorts and wraps a small sphere containing 0 around itself μ times.

Now, let's recall again that K is $(n-2)$-connected which means $\pi_i(K) = H_i(K) = 0$ for $i = 0, ..., n-2$ and $\pi_{n-1}(K) \cong H_{n-1}(K)$. Moreover, the Hurewicz map $h : \pi_n(K) \to H_n(K)$ is surjective if $n \geq 3$. Because of this, we can see that the only homology groups of K that are nontrivial are H_{n-1}, H_n (which are Poincaré dual).

Proposition 1.5 (Milnor)**.** *Suppose $n \neq 2$. Then K is homeomorphic to S^{2n-1} if and only if K has the homology of S^{2n-1}.*

The $n = 1$ case is not too hard. For $n \geq 3$, the proof is made easy in light of Smale's h-cobordism theorem which proved the Generalized Poincaré Conjecture. Theorem 1.2 says K is $(n-2)$-connected and so K is simply connected. We may construct an h-cobordism between K and S^{2n-1} before applying Smale's h-cobordism theorem.

Theorem 1.6 (Smale)**.** *Let $n \geq 5$ and let W be a compact $(n+1)$-dimensional h-cobordism between M and N in the category C=Diff, PL, or Top such that W, M and N are simply connected. This means W is homotopy equivalent to $M \times [0,1]$. Then, W is in fact C-isomorphic to $M \times [0,1]$ and can be the identity on $M \times \{0\}$.*

Here is an example of how this fails for $n = 2$.

Example 1.7. Let $n = 2$ and define $f : \mathbb{C}^3 \to \mathbb{C}$ as $f(z_1, z_2, z_3) = z_1^p + z_2^q + z_3^r$ where p, q, r are pairwise coprime. Then let $K = V_f \cap S^5$. This is a 3-manifold, called a Brieskorn sphere because it turns out that it is a homology 3-sphere. However, it always has nontrivial fundamental group isomorphic to the triangle group $\langle x, y, z : x^p = y^q = z^r = xyz = 1\rangle$.

For example, when $(p, q, r) = (2, 3, 5)$, $\pi_1(K)$ is isomorphic to $SL(2, \mathbb{F}_5)$ which has 120 elements. It is therefore, not homeomorphic to S^3. By Perelman and Hamilton's work, $\pi_1(K) = 0$ is enough to tell us that K is **diffeomorphic** to S^3 (simply because in dim 3, everything is smoothable and has only one smooth structure).

Example 1.8. We can obtain even more interesting behavior by studying such intersections. Let $f(t, w, x, y, z) = t^2 + w^2 + x^2 + y^3 + z^{6k-1}$. Then intersecting $f^{-1}(0) \cap S^9_\epsilon$ gives something that is topologically S^7. Letting $k = 1, ..., 28$ gives all the exotic smooth structures (and the standard one).

2 Picard-Lefschetz Theory

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic Morse function. This means that all its critical points are nondegenerate. Some of the results of smooth Morse theory hold in this context. Holomorphic Morse functions are dense among holomorphic functions. However, unlike in the smooth setting where we can define the index of a smooth Morse function by studying the Hessian, because of $\sqrt{-1}$, we can always write $f(z) = f(p) + \sum^{n+1} z_k^2$ locally around a critical point p.

Also, we cannot study the topology in the same way as over the reals because in $\mathbb{R}$, as we travel along, we are forced to pass through a critical value. But in $\mathbb{C}$, we can circumvent critical values. This feature actually gives some very interesting phenomena such as monodromy and vanishing cycles.

Anyways, because of the Morse chart result, let us assume that

$$f(z) = \sum_{k=1}^{n+1} z_k^2.$$

0 is the only singularity; as it turns out for $t \in \mathbb{C}^*$, the fiber $f^{-1}(t)$ is symplectomorphic to $T^*S^n = \{\xi + i\eta \in \mathbb{C}^{n+1} : |\xi|^2 = 1, \langle \xi, \eta \rangle = 0\}$ (as seen in the example above).

To see this, let's just take $t = 1$ and $z_k = x_k + iy_k$. Then $z_k^2 = x_k^2 - y_k^2 + 2ix_k y_k$ which means that $F_1 = f^{-1}(1) = \{z = x + iy : |x|^2 - |y|^2 = 1, \langle x, y \rangle = 0\}$. The map $F \to T^*S^n$ defined by $(x, y) \mapsto (x/|x|, |x|y)$ is a symplectomorphism where F has the symplectic form inherited from $\mathbb{C}^{n+1}$ and T^*S^n has the canonical symplectic form. It turns out all the other fibers other than over 0, are symplectomorphic to F.

Also, the S^n in T^*S^n shrinks as $t \to 0$, giving a singular fiber $f^{-1}(0)$. Thus, this S^n is a vanishinig cycle in the homology of the fibers. If we circle around 0 in $\mathbb{C}$ via a loop $\gamma(t)$ starting at F_1, we can define a family of diffeomorphisms $h_t : F_1 \to F_{\gamma(t)}$. Then $h_1 : F_1 \to F_1$ is a self-diffeomorphism (generalized Dehn twist?) and so induces a map on the homology level $h_1^* : H_*(F_1) \to H_*(F_1)$. Having the homotopy type of T^*S^n means that the only interesting homology is in the middle dimension: H_n. If we take homology with compact support, I think this allows us to consider the fibers as interesting cycles even though they are contractible. In this way, $m := h_1^* : H_n(F_1) \to H_n(F_1)$, called the monodromy, can be studied. It turns out, we can fully understand m by just studying the vanishing cycle.

Example 2.1. Let's take the example when $n = 1$. Then $f : \mathbb{C}^2 \to \mathbb{C}$ gives generic fibers T^*S^1, a cylinder. Let α be the zero section and β a fiber of the cylinder. Monodromy does a Dehn twist along α and so $\alpha \mapsto \alpha$ and $\beta \mapsto \alpha + \beta$ (it could be a minus instead, depending on orientation of the loop γ).

This discussion works for all sorts of complex manifolds V because it's a local discussion. If a holomorphic function has a bad singularity, such as when the Milnor number μ is large, a small perturbation to a Morse function splits the singularity apart into μ simple singularities. We can then define a monodromy group by looking at loops in $\pi_1(V \setminus \{z_1, ..., z_\mu\})$. However, to actually pinpoint where the singularities are is a difficult proposition.