Light Rays & Black Holes I

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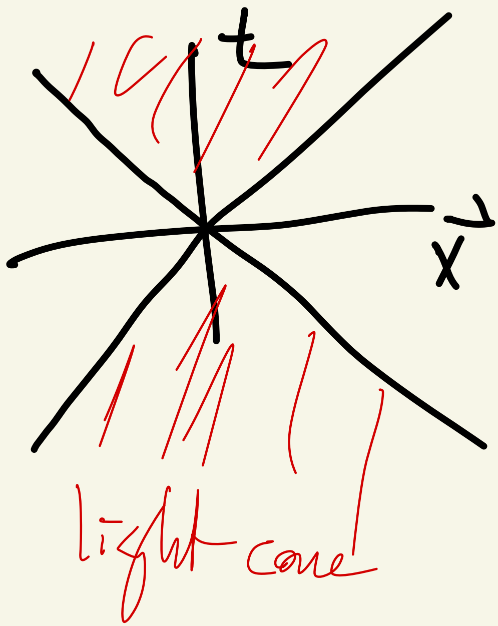
Causality in General Relativity

Let (M^{n+1}, g) be a spacetime w/ g a Lorentzian metric w/ signature $(- \underbrace{+ \cdots +}_{n})$. Then, for $x \in M$, $T_x M$ appears as



def: <u>Causal path</u>:

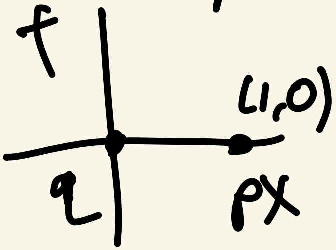
$\gamma : [0,1] \to M$ is causal b/w pts $q, p \in M$

if $\dot{\gamma}(t) \in \mathcal{L}_{\gamma(t)}$ (light cone at $\gamma(t)$)

$\forall t \in [0,1] \doteq I$.

thm: The space of causal paths b/w q & p, often denoted D_q^p; called the <u>causal diamond</u>, is compact.

nonexample. Let $(M^2, g = -dt^2 + dx^2)$



$\gamma_n(t) = (t, \sin(\pi n t))$.

Then γ_n has no convergent subsequence

However, if we take causal paths, this imposes the condition $\left|\frac{dx}{dt}\right| \leq 1$

$\Leftrightarrow$ the angle of γ w/ the x-axis is no more than $\pi/4$.

This bound on the derivative is enough to apply Arzela-Ascoli.

Thus, we have compactness in our example.

Philosophy: Causal paths are the key to understanding black holes.

Compare Special & General Relativity

SR	GR
◦ global time defined	◦ local time defined but global time is not
◦ deals w/ constant velocity frames of reference	◦ deals w/ forces; i.e. gravity & hence, accelerating frames of reference
◦ time is sort of singled out as a special dimension; the light cone is oriented around the time axis (even up to Lorentz transformations I think)	◦ time is not special (hence spacetime) but the metric does have signature $(-+++)$ so there is a special direction at any given pt.

Proper time of a path:

$$\tau : PM \longrightarrow \mathbb{R}$$

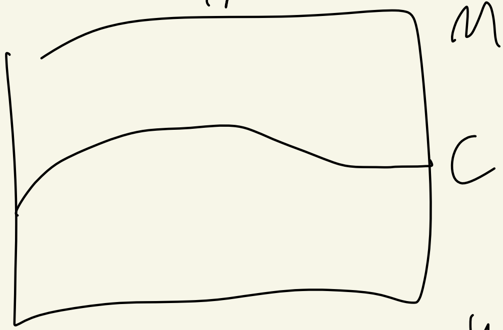
$$\tau(\gamma) = \int_0^1 ds \sqrt{\left(\frac{dt}{ds}\right)^2 - \left(\frac{dx}{ds}\right)^2} \quad \text{(parametrize } \gamma\text{)}$$

τ is upper semicontinuous. i.e. Say $\gamma_n \longrightarrow \gamma$. It isn't necessary that $\tau(\gamma_n) \longrightarrow \tau(\gamma)$. In fact, if $\tau(\gamma_n) \to \tau_0$, then $\tau_0 \leq \tau(\gamma)$.

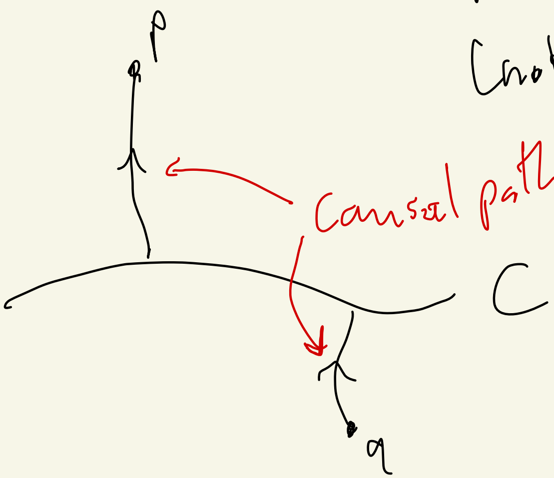
Cor. Twin Paradox. A & B are twins, A travels w/ acceleration & returns to earth while B travels along geodesic by remaining on earth. A is younger upon coming home, than B.

why?: There is a Lorentz frame of ref for B s.t. she is at rest but no such frame for A (due to acceleration).

Global hyperbolicity: $\exists$ complete hypersurface C in M s.t. all physics in M is determined by physics on C (initial conditions).



Think of all pts in M as being in the past or future of C (not both). Or on C itself.

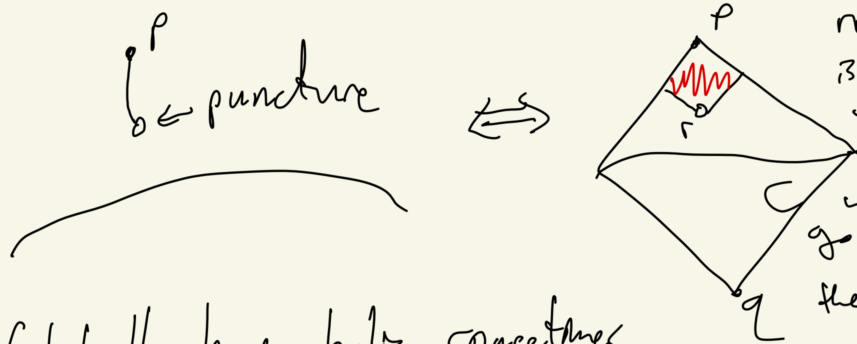


causal paths can be continued to meet C

Intuition: any signal that one observes at p must have arrived along a causal path from C.

C is sometimes called an <u>initial value surface</u>.

non examples:



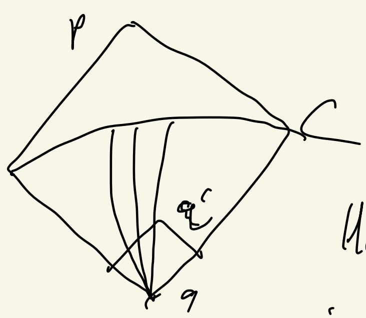
red region is murky; we don't know what's going on in that future.

Claim: Globally hyperbolic spacetimes have the property that spaces of causal paths w/ suitable conditions on endpoints are compact.

So we consider these spaces instead of Minkowski spacetime.

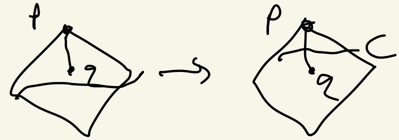
I think locally, a globally hyperbolic spacetime is Minkowski.

To prove the claim, you look locally, treating it as Minkowski:



First get convergent subseq to q'.
Repeat until you get to C.

How to get from q to p if both are in the future of C? Maybe flow C along to C'.



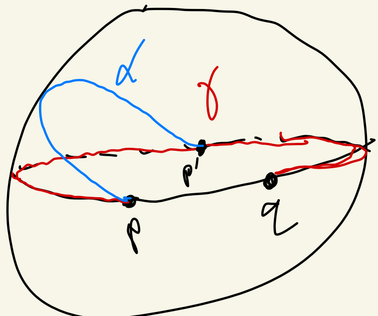
Witten goes anachronistically. Historically, one might begin w/ Penrose's results but Witten starts w/ Hawking's thm about Big Bang Singularity in traditional cosmology w/o inflation.

(easier b/c it only involves timelike geodesics; other applications require a subtler study of null geodesics).

(positive definite)
Riemannian Geodesics minimize distance on the small scale but not globally.
Geometry:

γ is a geodesic but doesn't minimize distance b/w p & q

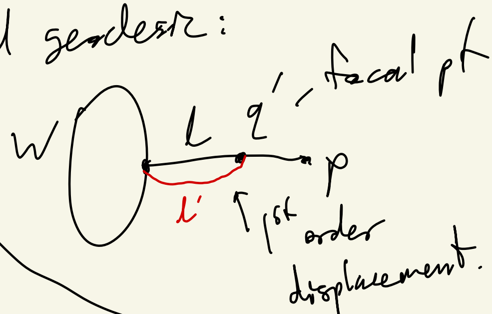
p' is the antipode of p.



If we follow α & run the rest of γ, we don't change length. So this new path, call it $\alpha+\gamma$, is not exactly a geodesic but solves the eqns in 1st order. If we smooth the kink, the length reduces in 2nd order

In the example, p', the antipode of p, is also called a <u>focal point</u>. In general, a focal pt is one where we can traverse from p to p' via a different geodesic, then go on to q.

In our setting, let W be an initial value hypersurface & ℓ an orthogonal geodesic:



Existence of focal pt for ℓ shows ℓ is <u>not</u> length minimizing. One can smooth out the kink to reduce length

Lorentz geometry.

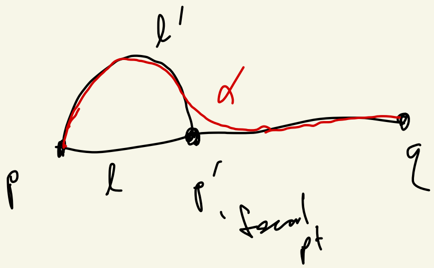
No analogous situation for spacelike geodesics which never max/minimize length. Perturb in spatial directions increase length (tend to)

Perturb in time directions decrease length

Cor. two pts w/ spacelike separation can be separated by an "everywhere" spacelike path that is arbitrarily short or long.

But for timelike geodesics, there is an analog.
Spatial fluctuations tend to reduce proper time.

$\therefore$ short timelike geodesics maximize proper time.
but the existence of focal pts will have similar properties
to the Euclidean signature case.



from p to p', one can take ℓ or ℓ' before going to q; no difference in length.
If one takes α, however
(α is a 2nd order smoothing of ℓ' + path to q)
it is shorter & has smaller proper time.

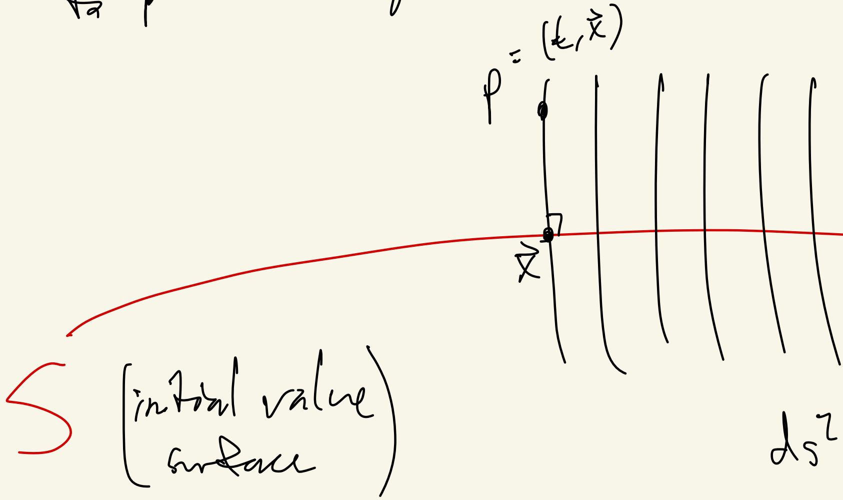
This is essentially the twin paradox →

Twin on earth takes ℓ to q. ← larger proper time
Twin traveling takes α. ← smaller proper time.

$\therefore$ traveling twin is younger due to the 2nd order smoothing = acceleration.

Claim: Focal pts are easy to come by b/c slight gravitational attraction can cause geodesics to eventually converge.

Goal: Prove timelike geodesics develop focal pts in order to prove a singularity thm.



orthogonal geodesics give coordinates

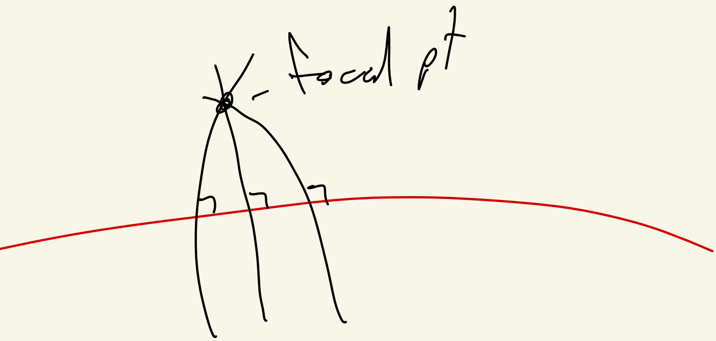
metric can be written as

$$ds^2 = -dt^2 + \sum_{i,j}^{d} g_{ij}\, dx^i dx^j$$

$g_{00} = -1$

$g_{0i} = 0$

Problem:



A suff condition for focal pts: $\det g_{ij} \longrightarrow 0$.

The Raychaudhuri eqn gives suff condition for

$\det g_{ij} \to 0 \Rightarrow \exists$ focal pts (or spacetime singularity) w/i a known proper time.

Eqn: $R_{00} = 8\pi G\left(T_{00} - \frac{1}{2} g_{00} T^{\alpha}_{\alpha}\right)$ just Einstein's eqn in the coord we defined w/ orthog geodesics

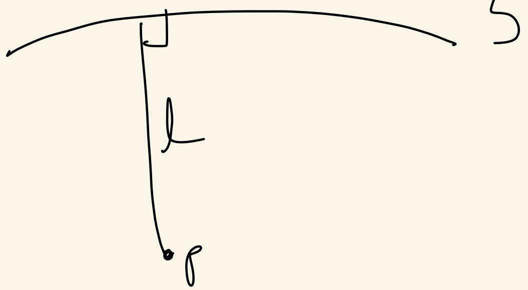
Apparently, we need a strong energy condition i.e. stress tensor is made of ordinary matter & radiation & the $\exists$ of focal pts is proved.

Hawking's Thm: Assume M is globally hyperbolic w/ hypersurface S & assume the strong energy condition. Then there is a Big Bang.

More precisely: If the local Hubble constant has a minimum value $h_{min} > 0$ on initial value surface S, then there is no point in spacetime with a proper time more than h_{min} to the past of S. (there's a beginning!)

So there's an upper bound on how long anything in the universe could have existed.

pf: 1.



Every pt p is connected by causal path of maximal proper time to S.

ℓ Does not have focal pt.

2. The Hubble assumption means the value

$$\frac{\dot{A}}{A} \leq -d\,h_{min}. \Rightarrow$$ any past going timelike orthogonal geodesic to S develops a focal pt within proper time at most $\frac{1}{h_{min}}$.

$\uparrow$ dimension

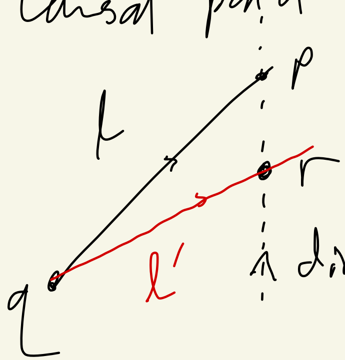
$A = \sqrt{\det g}$

$\therefore$ the time in which the universe extends into the past is bounded below.

i.e. there is a beginning. $\blacksquare$

Null geodesic: path that massless particles such as photons follow. They have proper time $= 0$ & are on the edges of light cones.

def: A causal path is <u>prompt</u> if there is no causal path from q to p which could have arrived sooner.

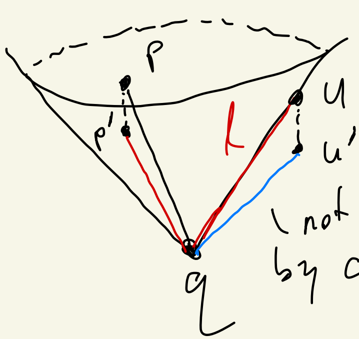


i.e. no l' which arrives at a pt r w/ same spatial coord as p but is in the past.

Basically, we immediately have that p is in the boundary of the causal future of q.

" all pts reachable from q by future-going causal paths.

notation: $J^+(q)$



Prompt paths should be unique, I think

In Minkowski spacetime, every null geodesic is prompt & every pt in $\partial J^+(q)$ (boundary of causal future) is connected to q by a prompt null geodesic.

In Lorenz signature spacetime, short segments of null geodesics are prompt but if continued, they may become non-prompt due to gravitational lensing.

e.g. Astronomers have observed the same supernova at different times b/c of how the light was bent by certain massive objects. The images that arrived later were not prompt.

If a geodesic has a tangent vector which is timelike, it cannot be prompt.

In other words, locally, light is always taking the fastest path.

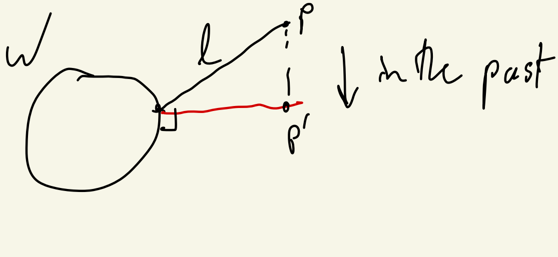
In Lorentz sig spacetime then, a null geodesic in order to be prompt, cannot have any focal pts.

In Minkowski space, any two pts which are null-separated can be mapped to any other two null separated pts by a Poincaré transformation. So there's no good notion of distance along null geodesics. Hence, the discussion w/ focal pts is the right notion to consider, instead of length.

Recap: ∘ For timelike geodesics, we can talk about length minimizing in terms of distance or focal pts.

∘ For null geodesics, we talk about "optimal" geodesics via focal pts.

We can talk about prompt null geodesics from an initial value surface W. They need to be orthogonal to W.



Claim: If W is spacelike, then for paths to be orthogonal to W, we need $\operatorname{codim} W \geq 2$. Why?

While not every null geodesic is prompt in Lorentz spacetime, it is true that every pt in $\partial J^+(q)$ is connected to q by a prompt null geodesic.

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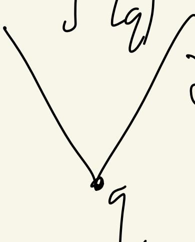
def: A set C is achronal if $\forall$ pairs $u, v \in C$, there is no timelike path in C connecting the two pts.

e.g. $\partial J^+(W)$ is achronal.

The boundary of the future of a closed set W is always a $d-1$ mfd w/o boundary though often not smooth.

e.g. $J^+(q)$



q

$\partial J^+(q) =$