Seiberg-Witten Invariants of Kähler Surfaces

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These notes arise from chapter 7 of John Morgan's *The Seiberg-Witten Equations and Applications to the Topology of Smooth 4-Manifolds*. I leave out many details and even important results. My main goal is to give enough background to understand some of the computations of section 7.4 in his book (here in the notes, that is section 4).

1 Equations Over a Kähler Manifold

Let X be a Riemannian 4-manifold with an orthogonal almost complex structure $J : TX \to TX$. Recall that this structure is a lifting from the group $SO(4)$ to $U(2)$. However because inclusion of $U(2) \subset SO(4)$ factors through $Spin^c(4)$, this lifting lifts our $SO(4)$ bundle to a $Spin^c(4)$ bundle.

$$\begin{array}{ccc} U(2) & \hookrightarrow & Spin^c(4) \\ \downarrow & \swarrow & \\ SO(4) & & \end{array}$$

Hence, though all Riemannian 4-manifolds have $Spin^c$ structures, if it additionally has an orthogonal almost complex structure, we get a very particular $Spin^c$ structure $\widetilde{P}_J$. The determinant line bundle of this structure is K_X^{-1}, the inverse of the canonical line bundle of the almost complex structure. Recall, that if we complexify the tangent bundle, we can split it into two parts using J: $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$. Then $K_X := \Lambda^{n,0}\, T^*_{\mathbb{C}}X := \Lambda^n(T^*)^{1,0}X$.

The spin bundles are given by

$$S^+(\widetilde{P}_J) = \Omega^0(X, \mathbb{C}) \oplus \Omega^{0,2}(X, \mathbb{C}), \quad S^-(\widetilde{P}_J) = \Omega^{0,1}(X, \mathbb{C}).$$

Moreover, if the metric is Kähler, then X is a Kähler surface and the Dirac operator for $\widetilde{P}_J$ and the natural holomorphic, hermitian connection on K_X^{-1} (I think Morgan means the Chern connection) is

$$\sqrt{2}(\bar{\partial} + \bar{\partial}^* 0 : \Omega^{)}(X, \mathbb{C}) \oplus \Omega^{0,2}(X, \mathbb{C}) \to \Omega^{0,1}(X, \mathbb{C}).$$

Any other $Spin^c$ structure $\widetilde{P}$ differs from $\widetilde{P}_J$ by tensoring with some $U(1)$-bundle $Q \to X$. Let $\mathcal{L}_0$ be the complex line bundle of Q. Then

$$S^+(\widetilde{P}) = S^+(\widetilde{P}_J) \otimes \mathcal{L}_0 = \Omega^0(X, \mathcal{L}_0) \oplus \Omega^{0,2}(X, \mathcal{L}_0), \quad S^-(\widetilde{P}) = S^-(\widetilde{P}_J) \otimes \mathcal{L}_0 = \Omega^{0,1}(X, \mathcal{L}_0).$$

Clifford multiplication and having a unitary connection A on $\mathcal{L}_0$ makes it so that our Dirac operator becomes $\sqrt{2}(\bar{\partial}_{A_0} + \bar{\partial}^*_{A_0})$ where $A_0^2 = A_{K_X} \otimes A$. Here, A_{K_X} is the Chern connection on K_X. This Dirac operator couples $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$ with the covariant derivative ∇_{A_0} on $\mathcal{L}_0$.

A spinor $\psi = (\alpha, \beta) \in \Omega^0(X, \mathcal{L}_0) \oplus \Omega^{0,2}(X, \mathcal{L}_0)$ and if it is harmonic, i.e. $\partial\!\!\!/_{A_0}\psi = 0$, this means $\bar{\partial}_{A_0}(\alpha) + \bar{\partial}^*_{A_0}(\beta) = 0$. What does this means for the curvature equation? After some computations, we can show that if ω is the Kähler form, then the curvature equation is equivalent to the following two equations:

$$(F_A^+)^{1,1} = \frac{i}{4}(|\alpha|^2 - |\beta|^2)\omega \tag{1.1}$$

$$F_A^{0,2} = \frac{\overline{\alpha}\beta}{2} \tag{1.2}$$

2 Holomorphic Description of the Moduli Space

Let us fix a connected Kähler surface X and a $Spin^c$ structure $\widetilde{P}$ whose determinant line bundle is $\mathcal{L}$ and we have $\mathcal{L}_0$ such that $\mathcal{L}_0^2 = K_X \otimes \mathcal{L}$. The **degree** of $\mathcal{L}$ is given by

$$\deg(\mathcal{L}) = \int_X c_1(\mathcal{L}) \wedge \omega.$$

Lemma 2.1. *Under the assumptions above (X is a connected Kähler surface, etc.), let (A, α, β) be a solution of the Seiberg-Witten equations for $\widetilde{P}$. If $\deg \mathcal{L} \leq 0$, $\beta = 0$. If $\deg \mathcal{L} \geq 0$, $\alpha = 0$. Moreover, A induces a holomorphic structure on $\mathcal{L}$ which in turn, induces a holomorphic structure on $\mathcal{L}_0$. With respect to this second holomorphic strucutre, α is a holomorphic section of $\mathcal{L}_0$ while $\bar{\beta}$ is a holomorphic section of $K_X \otimes \mathcal{L}_0^{-1}$.*

3 Evaluation for Kähler Surfaces

There is a natural way of orienting our moduli spaces. The details are in section 7.3. I will simply state an important proposition.

Theorem 3.1 (7.3.1)**.** *Let X be a Kähler surface with Kähler metric. Then:*

1. *If $\deg K_X < 0$, the only solutions to the Seiberg-Witten equations are reducible.*
2. *Let $\widetilde{P}_X$ be the $Spin^c$ structure determined by the complex structure. If $\deg K_X > 0$, then $SW(\widetilde{P}_X) = 1$ when the Seiberg-Witten invariant is computed with respect to the given Kähler metric.*

4 Computation for Kähler Surfaces

4.1 Surfaces of General Type

Let X be a minimal algebraic surface of general type. Algebraic here means projective. Being of **general type** means that its Kodaira dimension is 2; the characteristic property is that its canonical (complex) line bundle $K_X := \Lambda^{2,0} T^*X$ satsifies $K_X^2 := K_X \cdot K_X > 0$. What is meant here is that any complex line bundle produces a divisor, an element of $H_2(X, \mathbb{Z})$. Being a 4-manifold, it has an intersection form (which is Poincaré dual to cup product). The self-intersection of a divisor D is defined by taking a representative of D (which we can take to be an embedded complex curve C) and perturbing it slightly to some C' which is transverse but still homologous to C. This means $C \cap C'$ will be a finite, discrete set of points, by transversality. We sum up the intersections, counted with signs. That is the meaning of K_X^2.

Being a minimal algebraic surface means that K_X is **numerically effective** or **nef**. This means that for any **effective** divisor D of X, $K_X \cdot D \geq 0$. An effective divisor is one with only positive coefficients (in some basis).

Theorem 4.1 (7.4.1)**.** *Let X be a minimal algebraic surface of general type. Then for* ***any*** *Kähler metric, we have*

$$SW(\widetilde{P}) = \begin{cases} 1 & \widetilde{P} \cong \widetilde{P}_X; \\ (-1)^{(1+b_2^+-b_1)/2} & \widetilde{P} \cong -\widetilde{P}_X; \\ 0 & \text{otherwise.} \end{cases}$$

Some remarks: Recall that we need $b_2^+ - b_1$ to be odd in order to have nontrivial invariants. $(1 + b_2^+ - b_1)/2$ is denoted in Morgan's book with $\epsilon(X)$; in algebraic geometry, it is equal to $1+p_g(X) - q(X)$. Here, $p_g(X)$ is the **geometric genus** which is defined as $p_g(X) = h^0(K_X) = h^2(\mathcal{O}_X)$. $q(x)$ is the **irregularity** and is defined as $q(X) = h^0(\Omega_X^1) = h^1(\mathcal{O}_X)$.

Proof. Let ω be the Kähler form for any given Kähler metric. Thus, this ω might not be the Kähler form coming from some $\mathbb{C}P^N$. ω has a Poincaré dual in $H_2(X, \mathbb{Z})$ and hence, gives a divisor. It then makes sense to consider $K_X \cdot \omega \geq 0$ by the nef condition.

Now, for ω to give an **ample** divisor, we need to check the two criteria for a dvisor D to be ample: $D^2 > 0$ and $D \cdot E \geq 0$ for all divisors E. Since $\int_X \omega \wedge \omega = \int_X dVol/2! > 0$ and wedge product is Poincaré dual to the intersection, the first condition holds. Also, being a Kähler form, ω is symplectic and locally, appears as $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$; thus, $\omega \cdot E$ gives the volume of E which is positive. So ω gives an ample divisor. We use this fact in order to apply the following theorem:

Theorem 4.2 (Hodge Index Theorem, Algebraic Version)**.** *Let H be an ample divisor and D any divisor. Suppose $H \cdot D = 0$. Then $D^2 \leq 0$. If $D^2 = 0$, then $D \cdot E = 0$ for all divisors E.*

Let ω be our ample divisor. The minimal and general type condition make it so that K_X is big and ample. By Lazarsfeld (*Positivity I*), K_X is ample as well. Finally, two ample things intersect non-negatively. Thus, $\omega \cdot K_X \geq 0$. Now, suppose this is 0. Then $K_X^2 \leq 0$ which contradicts X being of general type. Thus, $\omega \cdot K_X > 0$ which means $\deg K_X := \int_X c_1(K_X) \wedge \omega > 0$. Theorem 3.1 tells us then that $SW(\widetilde{P}_X) = 1$. The evaluation of $SW(-\widetilde{P}_X)$ follows from Corollary 6.8.4 which is about an involution in the theory.

Our next task is to show that these are the only two $Spin^c$ structures on which SW is nonzero. Let $\widetilde{P}$ be a $Spin^c$ structure with non-negative formal dimension. This means that its determinant line bundle $\mathcal{L}$ satisfies $c_1(\mathcal{L})^2 \geq K_X^2 > 0$. This means that $c_1(\mathcal{L})^+$ is not a torsion class and thus, $[c_1(\mathcal{L})^+] = \frac{i}{2\pi}[F_A^+] \neq 0$. This means that we cannot have any reducible solutions because that would require $F_A^+ = 0$. If there are no reducible solutions, $\deg \mathcal{L} \neq 0$. As SW is symmetric under involution, $\deg(\det \widetilde{P}) \neq 0 \Leftrightarrow \deg(\det -\widetilde{P}) \neq 0$. So WLOG, assume $\deg \mathcal{L} < 0$. We now want to show that if $SW(\widetilde{P}) \neq 0$, $\widetilde{P} \cong \widetilde{P}_X$.

Of course, $SW(\widetilde{P}) \neq 0$ implies there is a solution. A solution implies the existence of a holomorphic structure $\bar{\partial}_A$ on $\mathcal{L}$ for which $\mathcal{L}_0 = \sqrt{K_X \otimes \mathcal{L}}$ has a nonzero holomorphic section. Let $L = PD(c_1(\mathcal{L}))$; the Poincaré dual of c_1; L is a divisor. The nonzero holomorphic section of $\mathcal{L}_0$ gives us an **effective** divisor $(K_X + L)/2$. K_X is nef so $K_X \cdot (K_X + L)/2 \geq 0$. Observe too that this implies $K_X^2 \geq -K_X \cdot L$. Since K_X has positive degree while $\mathcal{L}$ has negative degree, there exists a $t_0 \geq 0$ such that $\omega \cdot (K_X + t_0 L) = 0$. Note that t_0 may very well be in $\mathbb{Q}$. ω is ample so the Hodge Index Theorem implies that

$$0 \geq (K_X + t_0 L)^2 = K_X^2 + 2t_0 K_X \cdot L + t_0^2 L^2. \tag{4.1}$$

Since $K_X^2, L^2 > 0$ and $t_0 \geq 0$, then $K_X \cdot L < 0$.

Viewing this as a quadratic function $f(t)$ in t (replace the t_0), then its minimum can be found by taking the derivative with respect to t and setting it equal to 0. We find that when $t = -(K_X \cdot L)/L^2$, $f(t)$ is at a minimum and equals

$$K_X^2 - \frac{(K_X \cdot L)^2}{L^2}. \tag{4.2}$$

We would like to show that this expression, which is less than or equal to 0, is also greater than or equal to 0 and hence, equal to 0. From earlier, we have that $L^2 \geq K_X^2$ which means that $K_X^2 \cdot L^2 \geq (K_X^2)^2$. If we can show that $(K_X^2)^2 \geq (K_X \cdot L)^2$, we are done. But this is equivalent to showing $K_X^2 \geq K_X \cdot L$ which holds because $K_X \cdot L < 0$ and $0 < K_X^2$. Thus, Equation 4.2 equals 0.

Now, $K_X \cdot L < 0$ so $0 < -K_X \cdot L$. Moreover, we have a string of inequalities: $L^2 \geq K_X^2 \geq -K_X \cdot L > 0$. Therefore, in the expression on the RHS in 4.1, for it to be nonpositive, there are severe restrictions on what $t_0 \geq 0$ can be. For example, let's see if $t_0 = 2$. In that case,

$$K_X^2 + 4K_X \cdot L + 4L^2 \leq 0 \Longleftrightarrow K_X^2 + 4L^2 \leq -4K_X \cdot L.$$

But $L^2 \geq K_X^2$ which means that we have $5K_X^2 \leq -4K_X \cdot L$. But $K_X^2 \geq -K_X \cdot L$ so the previous inequality contradicts this fact and therefore, $t_0 \neq 2$. It's clear that if t_0 is any larger, we would continue to have a similar contradiction. However, if $t_0 = 1$, then $K_X^2 \leq -K_X \cdot L$ which doesn't contradict $K_X^2 \geq -K_X \cdot L$. Therefore, we need $t_0 = 1$ because any higher or lower, we would have a contradiction.

Hence, $\omega \cdot (K_X + L) = 0$ and $(K_X + L)^2 = 0$. The second part of the Hodge Index Theorem says that in this case, $(K_X + L) \cdot D = 0$ for all divisors D. This means that $K_X + L$ must be 0 (up to torsion) as the intersection form is unimodular. Then, since $\mathcal{L}_0 = \sqrt{K_X \otimes \mathcal{L}}$, $c_1(\mathcal{L}_0)$ is a torsion class. However, $\mathcal{L}_0$ has a non-trivial holomorphic section, so it is a holomorphic line bundle and lives in the Picard group. c_1 maps the Picard group into $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{Z})$ and $H^{1,1}$ does **not** have any torsion. Thus, we conclude that $c_1(\mathcal{L}_0) = 0$. These facts together imply it is **holomorphically** trivial. Then, $\xi = \mathcal{L}_0^2 = K_X \otimes \mathcal{L}$ is trivial which means $\mathcal{L} = K_X^{-1}$. This precisely means that the *Spin*^{c} structure in question is $\widetilde{P}_X$.

□

Corollary 4.3 (7.4.2)**.** *If X is a minimal algebraic surface of general type, then the Seiberg-Witten function is independent of the choice of Kähler metric.*

Remark: We already know that the SW function is independent of choice of metric when $b_2^+(X) > 1$ by Taubes' work but this theorem shows that it also holds for $b_2^+(X) = 1$.

Corollary 4.4. *Suppose that X and Y are two minimal Kähler surfaces of general type and that $f : X \to Y$ is an orientation-preserving diffeomorphism. Then $f^*(K_Y) = \pm K_X$.*

The *Spin*^{c} structures of a manifold M, if it admits any, are in 1-1 correspondence with $H^2(M, \mathbb{Z})$ as a set. Since X and Y are diffeomorphic, they have the same *Spin*^{c} structures and the same Seiberg-Witten invariant. Thus, $SW(f^*K_Y) \neq 0$ and thus, $f^*K_Y = \pm K_X$ by the theorem. I'm not sure why we need orientation-preservation.

In this proof, we made use of the Hodge Index Theorem. It's worth pointing out a consequence of the theorem:

Corollary 4.5. *Let X^n be a Kähler manifold. Then, we can define an intersection form on $H_n(X, \mathbb{Z})$ and thus, a signature σ. It turns out that $\sigma(X) = \sum_{p,q} (-1)^p h^{p,q}(X)$ where $h^{p,q} = \dim H^{p,q}(X, \mathbb{Z})$.*

An example application is to show that if X is a K3 surface, the signature is -16. The Hodge diamond is

$$\begin{matrix} & & h^0 & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,2} & & h^{2,1} & \\ & & h^4 & & \end{matrix} \quad = \quad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

Then the signature is $\sigma(K3) = 1 + 1 - 20 + 1 + 1 = -16$. Recall that a K3 is spin and spin 4-manifolds have signature divisible by 16 by Rokhlin's theorem.

It's also worth mentioning Kodaira's Embedding Theorem. A holomorphic line bundle is called **positive** if its curvature differential 2-form is, after multiplication with $i = \sqrt{-1}$, a positive definite $(1,1)$-form.

Theorem 4.6 (Kodaira's Embedding Theorem)**.** *Let X be a compact Kähler manifold, and L a holomorphic line bundle on X. Then L is positive line bundle if and only if there is a holomorphic embedding $\varphi : X \to \mathbb{P}$ of X into some projective space such that $\varphi^* \mathcal{O}_{\mathbb{P}}(1) = L^{\otimes m}$ for some $m > 0$.*

The theorem implies that a positive line bundle is an ample line bundle and conversely that any ample line bundle admits a Hermitian metric that makes it a positive line bundle. Also, the theorem says that if X admits a positive (holomorphic) line bundle, then X is projective. Furthermore, $\mathcal{O}_{\mathbb{P}}(1)$ has hyperplane sections in some abundance. Thus, a pullback by an embedding will yield a bundle with an ample amount of sections. Here, I'm using "ample" in the English sense to mean plentiful or sufficiently many.

4.2 Elliptic Surfaces

Let X be a minimal Kähler surface which is elliptic. We know what minimal means from before. An algebraic surface X is **elliptic** if there is a proper morphism $\pi : X \to C$ with connected fibers to an algebraic curve C such that almost all fibers are smooth curves of genus 1. A proper morphism is an analog to proper maps (preimage of compact is compact) in the complex analytic setting. Over an algebraically closed field, these fibers are elliptic curves, perhaps without a chosen origin.

All X with Kodaira dimension $\kappa = 1$ are elliptic though there are elliptic surfaces with $\kappa \neq 1$. When $\kappa = 1$, then the self-intersection of the canonical bundle is $K_X^2 = 0$. Morgan, when he refers to minimal elliptic surfaces in the following theorem, is assuming that $K_X^2 = 0$ and K_X is nef.

Theorem 4.7 (7.4.4)**.** *Suppose that X is a minimal Kähler **elliptic** surface and K_X is **not** a torsion class. Then $SW(\widetilde{P}_X) = +1$, $SW(-\widetilde{P}_X) = (-1)^{\epsilon(X)}$.*

Furthermore, if $\widetilde{P}$ is a Spin^c structure for X with $\mathcal{L} = \det \widetilde{P}$ and $SW(\widetilde{P}) \neq 0$, then the image of $c_1(\mathcal{L})$ in rational cohomology is a rational multiple of the image of K_X, with the multiple being between -1 and 1. If the multiple is ± 1, then the Spin^c structure is $\pm \widetilde{P}_X$.

Remark: I think the image of c_1 he refers to is from the map $H^2(X, \mathbb{Z}) \to H^2(X; \mathbb{Q})$. Any torsion vanishes after tensoring by $\mathbb{Q}$.

Proof. Our assumptions are that X is such that K_X is nef and not a torsion class and $K_X^2 = 0$. Let ω be any Kähler form; it is ample and thus, effective. K_X being nef means $K_X \cdot \omega \geq 0$. If $K_X \cdot \omega = 0$, then the Hodge Index Theorem tells us that $K_X^2 = 0$ implies that $K_X \cdot D = 0$ for all divisors D; i.e. K_X is a torsion class. This contradicts our assumption so $K_X \cdot \omega > 0$. Thus,

K_X has positive degree which allows us to apply Theorem 3.1 and Corollary 6.8.4 to conclude the values of $SW(\widetilde{P})$ and $SW(-\widetilde{P})$.

Now, let's fix a $Spin^c$ structure $\widetilde{P}$ for which $SW(\widetilde{P}) \neq 0$. Thus, there are solutions to the SW equations and the formal dimension of the moduli space is non-negative. Symmetry of the involution allows us to just consider the case when the determinant line bundle $\mathcal{L}$ is of non-positive degree. The moduli space dimension condition implies that $\mathcal{L}^2 \geq K_X^2 = 0$. If $\deg \mathcal{L} = \mathcal{L} \cdot \omega = 0$, the Hodge Index Theorem implies that $\mathcal{L}^2 \leq 0$ and so, is 0. This implies that $\mathcal{L}$ is a torsion class and its Poincaré dual, $c_1(\mathcal{L})$ is a torsion class. Our assumptions allow for such a situation.

Let us now assume the $\deg \mathcal{L} < 0$. This means that the holomorphic section α of $\mathcal{L}_0 := \sqrt{K_X \otimes \mathcal{L}}$ is nontrivial. Let $L := c_1(\mathcal{L})$ As in the proof for surfaces of general type, this holomorphic section and the fact that K_X is nef allows us to conclude that $K_X + L$ is an effective divisor and so $K_X \cdot (K_X + L) = K_X \cdot L \geq 0$.

Like the case of general type, there is a $t_0 > 0$ such that $\omega \cdot (K_X + t_0 L) = 0$. The Hodge Index Theorem tells us that $(K_X + t_0 L)^2 = 2t_0 K_X \cdot L + t_0^2 L^2 \leq 0$ and equals 0 if and only if $K_X + t_0 L$ is torsion. Since $K_X \cdot L \geq 0$ this means that $L^2 \leq 0$. We'll show that it equals 0.

Suppose $L^2 < 0$. Let $f(t) = (K_X + tL)^2 = 2tK_X \cdot L + t^2 L^2$. It has a minimum when $t_1 = -(K_X \cdot L)/L^2$ and $f(t_1) = -(K_X \cdot L)^2/L^2$. But $K_X \cdot L \geq 0$ and $L^2 < 0$ so $f(t_1) \geq 0$, contradicting that it is minimal at t_1. Thus, $L^2 = 0$ which implies that $2t_0 K_X \cdot L \leq 0$. $t_0 > 0$ and $K_X \cdot L \geq 0$ together imply that $K_X \cdot L = 0$ and thus, $(K_X + t_0 L)^2 = 0$, implying $K_X + t_0 L$ is torsion.

Now, K_X is nontrivial, even modulo torsion. Thus, when we do mod out by torsion, we have that $0 \neq K_X = -t_0 L$. K_X and L are both integral cohomology classes so $t_0 \neq 0 \in \mathbb{Q}$.

t_0 cannot be less than -1 or else $\mathcal{L}_0$ has negative degree which contradicts it having a non-trivial holomorphic section. If $t_0 = -1$, then $\deg \mathcal{L}_0 = 0$; having a nontrivial holomorphic section and zero degree implies that $c_1(\mathcal{L}_0) = 0$; thus$\mathcal{L}_0$ is holomorphically trivial and the $Spin^c$ structure is in fact $\widetilde{P}_X$. The case for when $\mathcal{L}$ has positive degree is established by symmetry. □

Corollary 4.8. *Suppose that X and Y are minimal Kähler surfaces, each of which is either of general type or is an elliptic surface with canonical class which is not a torsion cohomology class. Suppose that $f : X \to Y$ is an orientation-preserving diffeomorphism. Then $f^*(K_Y) = \pm K_X$.*

We end with a theorem about the blowups of surfaces of the last two types: general type or elliptic with nontorsion canonical class. We will also give some brief remarks about the Minimal Model Program afterwards to contextualize the theorem.

Theorem 4.9. *Suppose that X is a Kähler surface whose minimal model $\overline{X}$ is either a surface of general type or an elliptic surface with $K_{\overline{X}}$ not a torsion class. Suppose that the nontrivial fibers of $X \to \overline{X}$ are exceptional curves $E_1, ..., E_k$. Let the Kähler metric have a Kähler class ω of the form*

$$\omega = \overline{\omega} + \sum_{i=1}^{k} \epsilon_i [E_i]^*$$

where the $\epsilon_i \in \mathbb{R}$ are sufficiently small and positive and $[E_i]^*$ is the Poincaré dual to the ith exceptional curves. Suppose that $\widetilde{P}$ is a $Spin^c$ structure on X for which the Seiberg-Witten invariant is nontrivial. Then $\widetilde{P}$ is isomorphic to the tensor product of the pullback of a $Spin^c$ structure $\overline{P}$ on $\overline{X}$ which has nonzero Seiberg-Witten invariant with a $U(1)$-bundle whose 1st Chern class is of the form $\pm E_1 \pm ... \pm E_k$. Furthermore, $SW_X(\widetilde{P}) = \pm SW_{\overline{X}}(\overline{P})$.

4.3 Brief Remarks About the Minimal Model Program

The basic idea of the theory is to simplify the birational classification of varieties by finding, in each birational equivalence class, a variety which is "as simple as possible." This is variety is the **minimal model**. The precise meaning of this search has evolved; originally for surfaces, it meant finding a smooth variety X for which any birational morphism $f : X \to X'$ with a smooth surface X' is an isomorphism.

In the modern formulation, the goal of the theory is as follows. Suppose we are given a projective variety X, which for simplicity is assumed non-singular. There are two cases based on its Kodaira dimension:

1. $\kappa(X) = -\infty$. We want to find a variety X' birational to X, and a morphism $f : X' \to Y$ to a projective variety Y such that $\dim Y < \dim X'$ with the anticanonical class $-K_F$ of a general fibre F being ample. Such a morphism is called a Fano fibre space.

2. $\kappa(X) \geq 0$. We want to find X' birational to X, with the canonical class $K_{X'}$ nef. In this case, X' is a minimal model for X.

The question of whether the varieties X' and X appearing above are non-singular is an important one. Naturally, we hope that if we start with smooth X, then we can always find a minimal model or Fano fibre space inside the category of smooth varieties. However, this is **false**, and so it becomes necessary to consider singular varieties also. The singularities that appear are called terminal singularities.

Every irreducible complex algebraic curve is birational to a unique smooth projective curve, so the theory for curves is trivial. In the case of surfaces, Castelnuovo's Contraction Theorem essentially describes the process of constructing a minimal model of any surface.

Before stating the theorem, here is a definition: A (-1)-curve is a smooth rational curve $C \subset X$ with self-intersection -1. Rational means that C is isomorphic to $\mathbb{P}^1$; e.g. the cuspidal cubic is rational but of course, has a singularity, which is allowed. Any such curve must have $K_X \cdot C = -1$. This shows that **if the canonical class is nef then the surface has no** (-1)**-curves**. Thus, surfaces with nef canonical class are considered minimal.

Theorem 4.10 (Castelnuovo's Contraction Theorem). *Let X be a smooth projective surface over and C a (-1)-curve on X. There exists a morphism from X to another smooth projective surface Y such that the curve C has been contracted to one point P, and moreover this morphism is an isomorphism outside C; i.e. X/C is isomorphic to $Y/\{P\}$.*

This contraction morphism is sometimes called a **blowdown**, which is the inverse operation of blowup. The curve C is called an **exceptional curve** of the first kind. Exceptional divisors are the nontrivial fibers of $X \to X'$.

Castelnuovo's theorem implies that to construct a minimal model for a smooth surface, we simply contract all the (-1)-curves on the surface, and the resulting variety $\overline{X}$ is either a (unique) minimal model with $K_{\overline{X}}$ nef, or a ruled surface (which is the same as a 2-dimensional Fano fiber space, and is either a projective plane or a ruled surface over a curve). In the second case, the ruled surface birational to X is not unique, though there is a unique one isomorphic to the product of the projective line and a curve.