A New Look at the Jones Polynomial & QFT

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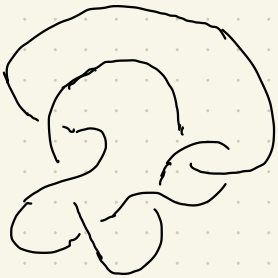
Jones Polynomial. (vertex model) Knots in 3-mfds

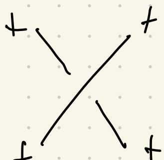
Given a knot projection to 2-dim plane w/ only simple crossings & only simple max/min of the height f^n.

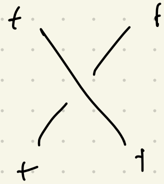


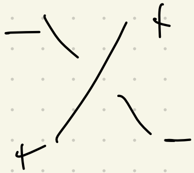
Remove the crossings & max/min. We're left w/ finitely many pieces. Label the pieces by $+/-$. Then introduce variable q & put local factors:

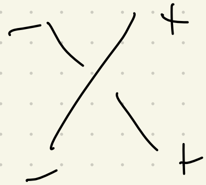
has, say, 15 crossings & max/min. Then there are 2^{15} possible labelings

e.g.

 $q^{1/4}$,

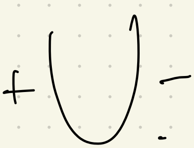
 $q^{-1/4}$

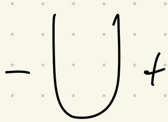
 $-q^{-1/4}$.

 0

Also have factors for creation & annihilation (from the max & min)

sounds physics-y like QM/QFT

 $iq^{-1/4}$

 $-iq^{1/4}$

$(i^2=-1)$

Get local factors for <u>all</u> possible labelings & multiply them all.

like statistical mech.

Then sum all the products together.

The sum is a Laurent polynomial in $q^{1/2}$.

We call this the <u>Jones polynomial.</u>

Claim: It is independent of the projection & thus a (framed) knot invariant.

Framing = trivialization of the normal bundle to the knot

The framing does depend on the projection but that just introduces powers of q in front: $q^n J(K)$

Jones poly can be defined for any Lie group G & representation R.

e.g. $G = SU(2)$, R = spin $\frac{1}{2}$ rep.

In 1988, Witten, w/ advice from Atiyah (also, a challenge from Atiyah, I think) found a description of Jones poly using 3-dim gauge theory on M^3. Let G = gauge gp, A = gauge field (connection)

Claim. The only gauge-invariant f^n of A that can be written as integration over M by some local expression (assuming no structure on M other than orientation) is the Chern-Simons functional: (or action) Need G-bundle to be trivial. If $G = SU(2)$, it always is.

$$CS(A) = \frac{1}{4\pi} \int_M Tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right).$$

It is an invariant, mod $2\pi\mathbb{Z}$

Partition Function of Chern-Simons Gauge Theory on M^3:

$$Z_k(M) = \frac{1}{vol} \int_{\mathcal{U}} \exp(ik\, CS(A))\, DA$$

$k \in \mathbb{Z}$

$\uparrow$ space of all connections

I suppose, M needs to be framed

To introduce a knot K: use holonomy of A.

Pick irrep R on K & define Wilson loop operator:

path ordering operator

$$W_R(K) = \mathrm{Tr}_R\, P \exp \underbrace{\oint_K A}_{\text{transport}}$$

def: Given M, K,

still need framing on M & K.

$$Z_k(M,K,R) = \frac{1}{\mathrm{Vol}} \int_{\mathcal{U}} \exp(ik\, CS(A)) \cdot W_R(K)\, \mathcal{D}A$$

Does this make sense? Apparently, the litmus test is renormalization to define it quantum mechanically. This has to do w/ counter terms & anomalies.

Comment: usual CS can't be renormalized. Is that true?

If $M = \mathbb{R}^3$, R - the 2dim rep, then $Z_k(M,k,R)$ is the Jones poly evaluated at $q = \exp(2\pi i/(k+2))$. Ranging over all k gives enough info to write down the Laurent poly.

Claim: $\mathbb{R}^3$ & S^3 are very similar in this situation. Just mod out gauge transformations which don't fix ∞. The answer differs by $\mathrm{vol}(G)$ which is indep of K.

But if $M \neq \mathbb{R}^3$ or S^3, it turns out that for most M, the Z_K's only depend on K & don't have natural continuations to f^{ns} of q, w/o losing some 3-dim symmetry; e.g. it may depend on the projection of the knot.

So the expectation values of the Wilson operators in $\mathbb{R}^3$ are special.

This means that perhaps this Chern-Simons approach is not very good or at least, doesn't generalize to other 3-mfds.

More on why CS is bad.

Let $G = U(1)$, persists in spacetime; have some sheeted material like paper

$$-\frac{1}{4}\int_{\text{Minkowski}} \overbrace{F_{\mu\nu}^2}^{= F \wedge *F} \, d^3x\, dt + \int_{\substack{\text{world}\\ \text{volume}\\ \text{of material}}} (\text{local expressions})\, d^3x\, dt$$

$$+ \int_{\text{boundary}} (\cdots)\, d^3x\, dt$$

I don't see the point here.

Seems like the boundary is an issue?

Even if our 3-mfd W has no boundary, A is a $U(1)$ conn,

$$I = \frac{1}{4\pi}\int_W \varepsilon^{ijk} A_i \partial_j A_k \, d^3x \text{ is defined mod } 2\pi$$

But we may get something better defined:

Use fact that all closed 3-mfds are the boundary of some 4-mfd. So say M^4 ; $\partial M = W$.

Let $F_{ij} = \partial_i A_j - \partial_j A_i$. Replace I w/

$$\hat{I} = \frac{1}{4\pi}\int_M \varepsilon^{ijkl} F_{ij} F_{kl} \, d^4x.$$

This is completely gauge invariant & is well-defined.

Moreover: $\hat{I} \equiv I \pmod{2\pi}$.

⋮

Upshot: If we find the right 4-d TQFT, we can get a more satisfactory explanation of why the Jones poly isn't limited to integers k but extend to q.

Challenges; we have an identity:

$$\int_W \operatorname{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) = \int_M \operatorname{Tr}\left(F \wedge F\right)$$

the RHS is indep of metric but is "too degenerate"

Seems the idea is: A physical supersymmetric theory is strongly coupled & is hard to understand directly. It is constrained by an associated topological field theory.

I think physical theory means we have a metric, topological means it is independent of metric.

Some work done to understand the Chern-Simons path integral which makes analytic continuation evident.

- Khovanov homology (categorification of the Jones polynomial)
- Volume conjecture