The Story of Hamiltonian Floer Homology

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These notes are compiled mainly from Audin and Damien's *Théorie de Morse et homologie de Floer*. Hamiltonian Floer homology aims to answer one rendition of Arnold's conjecture: The number of periodic solutions of period 1 of a nondegenerate time-dependent Hamiltonian system on a compact $2n$-symplectic manifold (W, ω) is greater than or equal to the sum

$$\sum_i \dim HM_i(W; \mathbb{Z}_2).$$

1 Basics for Setting up the Conjecture

Let us first consider an autonomous Hamiltonian function $H : W \to \mathbb{R}$. The Hamiltonian vector field X_H of H is defined by $\omega(Y, X_H) = dH(Y)$. More concisely, $i_{X_H}\omega = -dH$. Since ω is nondegenerate, one can immediately see that x is a critical point of H if and only if $X_H(x) = 0$. Moreover, we have the following proposition:

Proposition 1.1. *The time t flow of a Hamiltonian vector field is a diffeomorphism that preserves the symplectic form. It is called a* ***Hamiltonian diffeomorpshim****.*

Proof. Recall that if η is a k-form, the Lie derivative has a simple form: $\mathcal{L}_X\eta = di_X\eta + i_X d\eta$. Also,

$$\begin{aligned}\frac{d}{dt}(\psi^t)^*\omega &= \lim_{h\to 0}\frac{(\psi^{t+h})^*\omega - (\psi^t)^*\omega}{h}\\ &= (\psi^t)^*\lim_{h\to 0}\frac{(\psi^h)^*\omega - \omega}{h}\\ &= (\psi^t)^*\mathcal{L}_{X_H}\omega.\end{aligned}$$

The last line is simply the definition of the Lie derivative. Then we have

$$\begin{aligned}\frac{d}{dt}(\psi^t)^*\omega &= (\psi^t)^*\mathcal{L}_{X_H}\omega\\ &= (\psi^t)^*(di_{X_H}\omega)\\ &= (\psi^t)^*(-ddH) = 0.\end{aligned}$$

Thus, this family of pullbacks is independent of time. Since $\psi^0 = \mathrm{id}$, $(\psi^t)^*\omega = (\psi^0)^*\omega = \omega$ for all t. □

We now consider $H : \mathbb{R} \times W \to \mathbb{R}$, the time-dependent Hamiltonian; we have a family of Hamiltonians H_t. Then we can consider a family of Hamiltonian vector fields $X_t := X_{H_t}$. Though we don't get a family of flows, we still have a family of isotopies; in fact, they are symplectomorphisms ψ^t such that

$$\frac{d}{dt}\psi^t = X_t \circ \psi^t \text{ and } \psi^0 = \mathrm{id}\,.$$

What is missing: this family doesn't necessarily satisfy the group law.

Now, for both the autonomous and time-dependent Hamiltonian, the associated differential system is:

$$\dot{x}(t) = X_t(x(t)).$$

Obvious but important point: If $x(t)$ is a periodic solution to this differential system with period 1 ($x(0) = x(1)$), then $x(t)$ corresponds to a fixed point of the diffeomorphism ψ^1.

Definition 1.2. *A periodic solution $x(t)$ to the Hamiltonian system is called **nondegenerate** if the differential of ψ^1 at $x(0)$ does not have eigenvalue 1.*

If we can prove that there are finitely many such fix points for ψ^1, then there are finitely many periodic solutions to the Hamiltonian system. The way to prove this is the following: let $\Delta = \{(x, x) : x \in W\}$ be the diagonal of $W \times W$ and $\Gamma(\psi^1) = \{(x, \psi^1(x)) : x \in W\}$ be the graph of ψ^1. Then the intersection points of Δ and Γ represent the fixed points. If the periodic solutions are all nondegenerate, then the intersection is transverse. Thus, codim Δ+codim Γ = codim $(\Delta \cap \Gamma)$ which is $2n + 2n = 4n$. So the dimension of the intersection, which is a submanifold, is zero. Since $W \times W$ is compact, there must be finitely many such points and therefore, finitely many periodic solutions.

Two assumptions we make about (W, ω) moving forward:

1. For every smooth $f : S^2 \to W$, $\int_{S^2} f^*\omega = 0$. This assumption is often denoted $\pi_2(\omega) = 0$.
2. For every smooth $f : S^2 \to W$, there exists a symplectic trivialization of f^*TW. This means that $c_1(f^*TW) = 0$ for all such f. Here c_1 is the first Chern class.

By the way, recall that isomorphism classes of rank $2k$ symplectic vector bundles over a manifold M are in 1-1 correspondence with homotopy classes of maps $M \to BSp(2k)$. $BSp(2k)$ is the classifying space for symplectic vector bundles. However, since $Sp(2n)$ deformation retracts to $U(n)$, symplectic and complex vector bundles have the same classifying space. Therefore, symplectic bundles are equivalent to complex vector bundles and every symplectic manifold is an almost complex manifold. c_1 classifies complex line bundles.

2 Background on the Arnold Conjecture

This section is by Liviu Nicolaescu. Let's consider a trivial example from a survey article written by Arnold in the late 80s.

Consider T^*S^1, the cotangent bundle of S^1. We can identify it with the product $S^1 \times \mathbb{R}$ since its orientable and there are only two real line bundles over S^1 up to isomorphism (as given by the first Stiefel-Whitney class. The other one is the Möbius bundle). The obvious coordinates on one chart of this cylinder are (θ, t). Like any cotangent bundle, T^*S^1 carries a symplectic structure (I believe $\omega = dt \wedge d\theta$), and in this case, any curve on this symplectic manifold is a Lagrangian submanifold. There are different types of curves, however.

The curves $C_\tau := \{t = \tau\}$, ($\tau \neq 0$ a constant) are disjoint from the zero section and are deformations of the zero section via the symplectic flow $(\theta, t) \mapsto \Phi_\tau(\theta, t) = (\theta, t + \tau)$; i.e. $\Phi_\tau^*\omega = \omega$. The vector field giving rise to this flow is rather boring: at each point of the cylinder $S^1 \times \mathbb{R}$, assign a unit vector pointing along the $\mathbb{R}$ direction.

Consider next a smooth function $\theta \mapsto f(\theta)$. Its differential is a section of T^*S^1, and its graph $\Gamma_{df} = (\theta, f'(\theta))$ intersects the zero section along the critical points of f.

The Lagrangian submanifold Γ_{df} is a rather special deformation of the zero section: it is a **Hamiltonian deformation**. The points of intersection of Γ_{df} with the zero section correspond to the periodic orbits of the Hamiltonian deformation.

Why is this fascinating? Certain pairs of Lagrangian subspaces intersect in more points than predicted by topology alone, which is in itself **an indication that symplectic topology is a bit more rigid than smooth topology alone.** What I mean by this is that a Hamiltonian diffeomorphism is isotopic to the identity and so the Lefschetz number (from the Lefschetz Fixed Point Theorem) predicts a lower bound sum with alternating signs rather than just the sum: $\chi(M) = \sum_k (-1)^k b_k$. Usually, we need to consider the index of the fixed points but since we're dealing with Hamiltonian diffeomorphisms, the index of each fixed point is ± 1. Mark says that the Poincaré-Hopf index is equal to $(-1)^{CZ}$, where CZ is the Conley-Zehnder index.

Otherwise, on S^2, we could come up with a vector field which has only one fixed point of index $2 = \chi(S^2)$ in the following way. Consider some translation of $\mathbb{R}^2$ and extend this to S^2; the point at infinity is the fixed point. This map is homotopic to the identity but it must be degenerate; My guess is that if we look at the graph of it in $S^2 \times S^2$ and consider how it intersects the diagonal, there will be some sort of order 2 tangency (not transverse). Also, this map is not volume preserving.

How does the above trivial example fit the general picture? A Lagrangian submanifold L of a symplectic manifold has a tubular neighborhood symplectomorphic to T^*L. Thus, the case of cotangent bundles can be viewed as local situations of the more general cases of Lagrangian submanifolds and their Hamiltonian perturbations. Given a Hamiltonian flow Φ_t on a symplectic manifold X, the graph of the time 1-map is a Lagrangian submanifold in $X \times X$. As we saw above, its fixed points correspond to the intersection of the graph with the diagonal in $X \times X$, which is another Lagrangian submanifold. Thus the problem of intersection of Lagrangian submanifolds contains as a special case the problem of existence of periodic solutions of hamiltonian systems.

Leaving aside the mysterious rigidity of symplectic topology alluded to above, the problem of existence of periodic orbits of Hamiltonian systems has fascinated many classics, such as Poincaré, because of its obvious connection to the many body problem and the philosophical question: does the history of our planetary system repeat itself?

3 Floer Theory

Conley and Zehnder took inspiration from Morse theory and was able to resolve the conjecture for all even dimension tori. The genius of Andreas Floer (1956-1991) comes from combining the variational approach of Conley and Zehnder with Gromov's elliptic methods. The outline of the proof is given here.

1. Consider the space of free smooth contractible loops in W: $\mathcal{L}W$; it is in fact a Banach manifold. We're looking for periodic solutions so it makes sense to consider loops. Critical points are solutions in the autonomous case so it makes sense to consider contractible loops. A tangent vector at a point $x \in \mathcal{L}W$ is a vector field. Formally, consider $x : S^1 \to W$ and the pullback bundle x^*TW. Thus, a tangent vector to x is a section of x^*TW.

2. Define an action functional $\mathcal{A}_H$ on $\mathcal{L}W$ whose critical points (loops) are the solutions to the time-dependent Hamiltonian system. The action functional is well-defined because $\pi_2(\omega) = 0$.

3. Fix a compatible almost complex structure J for W. We get the Riemannian metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ on W which induces a metric $\langle \cdot, \cdot \rangle$ on $\mathcal{L}W$. We can then study the trajectories of the negative gradient of $\mathcal{A}_H$. These trajectories are smooth maps $u : \mathbb{R} \times S^1 \to W$ (a path of loops). They are also solutions to a PDE called the Floer equation. Thus, we can define an energy $E(u)$ for each u which is nonnegative and equals

zero if and only if $u(s,t)$ is independent of s. Show that if $E(u) < \infty$, $\lim_{s\to\pm\infty} u(s,t)$ are critical points of $\mathcal{A}_H$.

4. In the setup till now, one important fact we need comes from Gromov: let $\mathcal{M}$ be the space of finite energy, smooth contractible solutions of the Floer equation. Under the assumption $\pi_2(\omega) = 0$, $\mathcal{M}$ is compact. If we don't have this assumption, a phenomenon called "bubbling" appears.

5. We define a chain complex analogous to the Morse complex with these critical points. The Maslov index serves this purpose.

6. The differential ∂ for the complex is defined from $-\nabla\mathcal{A}_H$; it sends a critical point along a trajectory to another criticial point (when the trajectory has finite energy). As in the Morse case, we need a way to count the trajectories.

7. Also as in the Morse case, we'll like the space of trajectories to be a manifold and we'll also need genericness properties such as the Smale property. In the Morse case, the Smale property was for all stable and unstable manifolds to intersect transversally. We may thus need to consider a different vector field. It suffices to perturb the gradient slightly by perturbing the metric via the almost complex structure J.

8. We need $\partial^2 = 0$. Just as in the Morse case, we'll need a gluing property to prove this.

9. We show that the homology defined from this Floer complex is independent of the functional and vector field.

10. By choosing a C^2-small Hamiltonian and relying on the independence of the Floer homology, we find that Floer homology coincides with Morse homology. Thus, we get analogous results to the Morse inequalities.