$c_1^2(L)$ of the Determinant Line Bundle of the Standard $Spin^c$ Structure on an Almost Complex 4-Manifold

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Suppose we have a closed almost complex 4-manifold (X, J). This means we have a lifting of the frame bundle F on the tangent bundle, a $SO(4)$-bundle, to a $U(2)$-bundle. However, the natural injective morphism of $U(2) \to SO(4) \cong (SU(2) \times SU(2))/\{\pm \mathrm{id}\}$ factors through $Spin^c(4) \cong (SU(2) \times SU(2)) \times_{\mathbb{Z}_2} U(1)$; i.e. we get the natural morphism is a composition $U(2) \to Spin^c(4) \to SO(4)$. call the first map h. Thus, having a lifting F to a $U(2)$-bundle automatically gives a $Spin^c$ lifting as well. Call this lift P.

Now, we have a commutative diagram:

$$\begin{array}{ccc} U(2) & \xrightarrow{h} & Spin^c(4) \\ {\scriptstyle \det}\downarrow & \swarrow {\scriptstyle \det} & \\ U(1). & & \end{array}$$

This tells us that the determinant map is compatible with h. We can conclude then that the determinant line bundle of (X, J), that is, $K_X = \Lambda^2(T^*)^{1,0}X$ is dual to the determinant line bundle of P but they'll have the same 1st Chern class. Recall that $\det P$ can be constructed using classifying spaces: let $g : Spin^(V) \to S^1$ be the morphism that has $Spin(V)$ as kernel and $\tilde{g} : BSpin^C \to BS^1$ be the induced map. Let $f : X \to BSpin^c$ be a map representing its homotopy class such that $f^*ESpin^c \cong P$. Then $\det P \cong (\tilde{g} \circ f)^* ES^1$. Thus, $K_X \cong \det P$ and so they have the same 1st Chern class. Thus, let's see what $c_1^2(K_X)$ is.

Lemma 0.1. *The first Pontrjagin number $p_1 = c_1^2 - 2c_2$.*

Proof. Let E be a real vector bundle over a $4k$ manifold. We define the Pontrjagin classes using Chern classes in the following way: $p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C})$. The total Pontrjagin class is $p(E) = 1 + p_1(E) + p_2(E) + ... = c(E \otimes \mathbb{C}) = 1 + c_1(E \otimes \mathbb{C}) + c_2(E \otimes \mathbb{C}) + ...$

When E is already a $\mathbb{C}$-vector bundle, then $c(E \otimes \mathbb{C}) = c(E \oplus \overline{E})$. Then in this case, $p(E) = c(E \oplus \overline{E}) = (1 + c_1(E) + c_2(E) + c_3(E) + ...)(1 - c_1(E) + c_2(E) - c_3(E) + ...)$ Now, $p_1(E) = -c_2(E \otimes \mathbb{C})$ which only requires us to look at rank 4 terms in the above. So let's focus our attention to $(1 + c_1 + c_2)(1 - c_1 + c_2) = 1 - c_1 + c_2 + c_1 - c_1^2 + c_1c_2 + c_2 - c_1c_2 + c_2^2$. Taking only the rank 4 terms, we get $p_1 = -(2c_2 - c_1^2)$. □

Fact: For a 4-manifold, $p_1 = 3\sigma$ (signature). Also, for a manifold of complex dimension 2, $c_2 = \chi$ (Euler characteristic). Putting this together, $c_1^2(K_X) = 2\chi(X) + 2\sigma(X)$.

With this observation, one sees that for such a $Spin^c$ structure coming from ACS J, the dimension of the moduli space of Seiberg-Witten solutions is actually 0. Let's call the line bundle of the $Spin^c$ structure arising from J, the standard one S. If we twist the $Spin^c$ structure giving rise to S, we tensor it by a line bundle L and $c_1(S \otimes L) = c_1(S) + 2c_1(L)$ (the 2 comes from the fact that the determinant map is like squaring). Thus, we can compute the 1st Chern classes of all other $Spin^c$ structures if we know about the bundle we twist by. This is true even if we're not on an almost complex manifold but then there is no canonical reference. Also note, when we reduce mod 2, the $2c_1(L)$ part vanishes and so we still have $c_1(S \otimes L)$ mod $2 = w_2(X)$