Definitions and Equations from Hamiltonian Floer Theory

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The Hamiltonian Equation

- Let $\mathcal{L}M$ be the space of contractible loops in symplectic manifold (M, ω). That is, the space of contractible maps $x : S^1 \to M$. If $H : M \times \mathbb{R} \to \mathbb{R}$ is a time-dependent Hamiltonian, we are able to define Hamiltonian vector fields X_t as follows: $\iota_{X_t}\omega = -dH_t$. This defines for us a family of isotopies, φ_t. The Hamiltonian system is an ODE:

$$\dot{x}(t) = X_t(x(t))$$

The Action Functional

- The action functional is defined on $\mathcal{L}M$. Let $\bar{x} : D^2 \to M$ be an extension of the contractible loop $x : S^1 \to M$.

$$\mathcal{A}_H : \mathcal{L}M \to \mathbb{R};\ \mathcal{A}_H(x) = -\int_{D^2} \bar{x}^*\omega + \int_0^1 H_t(x(t))\, dt.$$

- The differential of the action functional. A tangent vector $Y(t)$ to a loop x is a vector field along the loop. i.e. a section of the pullback bundle x^*TM.

$$(d\mathcal{A}_H)_x(Y) = \int_0^1 \omega(\dot{x}(t) - X_t(x(t)), Y(t))\, dt.$$

  Note that because ω is nondegenerate, the differential equals 0 precisely when $\dot{x}(t) = X_t(x(t))$; i.e. x is a solution to the Hamiltonian equation.

The Floer Equation

- Let us define a metric on M by fixing an almost complex structure J compatible with ω. Then $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ extends to a metric on $\mathcal{L}M$ by integration. A trajectory of $\nabla \mathcal{A}_H$ is a map $u : \mathbb{R} \times S^1 \to M$ satisfying the Floer equation, a PDE. It is convenient, for example, to consider an equation in terms of an operator Δ acting on all functions $f : M \to \mathbb{R}$ and look for solutions f such that $\Delta f = 0$.

$$\mathcal{F}u = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla_u H_t = 0$$

- We care about, in particular, solutions of the Floer equation with **finite energy**. Energy is defined as follows:

$$E(u) = \int_{-\infty}^{\infty}\int_0^1 \left\|\frac{\partial u}{\partial s}\right\|^2 dt\, ds.$$

- The linearization of a differential equation along a solution u means that we consider first the operator and then take the differential at a solution u. In our case, linearizing the Floer map along a solution (having chosen some trivializations on u^*TM) u yields:

$$L_u Y := (d\mathcal{F})_u Y = \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y.$$

  $S(s,t)$ is a linear operator which tends to symmetric operators as $s \to \pm\infty$. Therefore, $\lim_{s\to\pm\infty} \partial S/\partial s = 0$.

- The adjoint of L_u is denoted L_u^* and defined in the usual way: $\langle L_u Y, Z\rangle = \langle Y, L_u^* Z\rangle$. Explicitly, we have a formula:

$$L_u^* Z = -\frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S^t Z.$$

  This is easy to see. We should consider this in the setting of distributions since we're dealing with Sobolev spaces. Then

$$\left\langle \frac{\partial Y}{\partial s}, Z \right\rangle = -\left\langle Y, \frac{\partial Z}{\partial s} \right\rangle.$$

  With $J_0 \frac{\partial}{\partial t}$, the partial derivative introduces a minus sign but so does J_0 since $J_0^t = -J_0$.

The Morse Equation

- Let $f : M \to \mathbb{R}$ be a Morse function. A trajectory $u : \mathbb{R} \to M$ of ∇f satisfies

$$\frac{du}{ds} + \nabla f(u(s)) = 0.$$

  In the case of an autonomous C^2 small Hamiltonian H, we can show that its periodic orbits are constant and that H is Morse. Then, with a fixed almost complex structure J, we have a way to define $\nabla H = -JX_H$. Our equation becomes:

$$\frac{du}{ds} - JX_H(u(s)) = 0.$$

- The linearization of this ODE along a solution u is

$$L_u Y = \frac{dY}{ds} + A(s)Y$$

  where $A(s) \to \mathrm{Hess}_x f$ as $s \to -\infty$ and $A(s) \to \mathrm{Hess}_y f$ as $s \to +\infty$.

- The adjoint of L_u is L_u^* defined by $L_u^* Z = -\frac{dZ}{ds} + A^t Z$.

Some Definitions

- A solution u to the Floer equation is "somewhere injective" if for any fixed t_0, whenever $u(s,t_0) = u(s_1,t_0)$, $s = s_1$. So, it is perfectly possible for $u(s_0,t_0) = u(s_1,t_1)$ with $t_0 \neq t_1$. It is a result that any solution u that is not simply an orbit $x(t)$ is somewhere injective.