Energy - Momentum Tensor (p. 49)
(or Stress - Energy tensor) $g = \mathrm{diag}(-1,1,1,1)$

Given $\varphi : (\mathbb{R}^4, g) \to \mathbb{R}$, the energy-momentum tensor is defined as

$$T = d\varphi \otimes d\varphi - \frac{1}{2}\left(g^{-1}(d\varphi, d\varphi) - m^2\varphi^2\right) g$$

(it's symmetric)

Its components: $T_{\alpha\beta} = T_{\beta\alpha}$

$$T_{\alpha\beta} = \varphi_{,\alpha}\varphi_{,\beta} - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta}\varphi_{,\gamma}\varphi_{,\delta} - \frac{1}{2} m^2 \varphi^2 g_{\alpha\beta}$$

Then $J^{\alpha}_{energy} = -T^{\alpha}_{t} \equiv -g^{\alpha\beta} T_{t\beta}$

& energy density $= T^{tt}$

$$J^{\alpha}_{momentum} = -T^{\alpha}_{i} \equiv -g^{\alpha\beta} T_{i\beta}, \quad i = 1,2,3$$

Momentum density in direction i is $-T^{ti}$

Conservation of energy & momentum is encoded in an identity:

$$g^{\beta\gamma}\partial_\gamma T_{\alpha\beta} = \varphi_{,\alpha}(\Box - m^2)\varphi$$

where $\Box\varphi = g^{\alpha\beta}\varphi_{,\alpha\beta}$.

So if $(\Box - m^2)\varphi = 0$, then

$$g^{\beta\gamma}\partial_\gamma T_{\alpha\beta} = 0$$

i.e. the divergence of the energy-momentum tensor vanishes

Note: Change of reference frame mixes up energy & momentum. So one often says, "Conservation of energy-momentum", not just one of these.

A symmetry $f: \mathbb{R}^4 \to \mathbb{R}^4$ is a diffeo s.t. if $\hat{\varphi} = \varphi \circ f$, then the Lagrangian is preserved.

So if
$$\mathcal{L}(x, \hat{\varphi}, \partial\hat{\varphi}) = \mathcal{L}(x, \varphi, \partial\varphi)$$
then the Lagrangian is preserved. There is a more general notion of symmetry called divergence symmetry.

Since, if $\hat{\mathcal{L}} = \mathcal{L} + (\text{div term})$, then

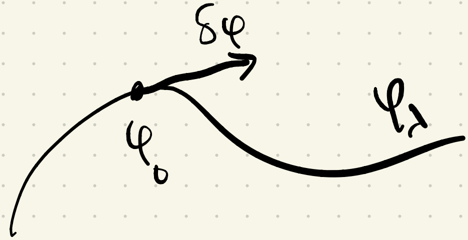
$\mathcal{E}(\hat{\mathcal{L}}) = \mathcal{E}(\mathcal{L})$, diffeos which change the Lagrangian by a divergence term are also considered to be symmetries.

we typically study 1-param families of symmetries, say $f_\lambda : \mathbb{R}^4 \to \mathbb{R}^4$ & let $\varphi_\lambda \doteq \varphi \circ f_\lambda$. Then

we want $\frac{d\mathcal{L}(x, \varphi_\lambda, \partial_x \varphi_\lambda)}{d\lambda} = 0$.

Infinitesimal Symmetry (a vector field) p.55

Space of fields



— path of fields defined by continuous symmetries

$$\delta\varphi \doteq \left.\frac{d\varphi_\lambda}{d\lambda}\right|_{\lambda=0}.$$

This $\delta\varphi$ is an infinitesimal symmetry.

View it as a vect field on the space of fields

Since Lagrangian densities which differ by a divergence term gives the same Euler-Lagrange eqns, then we expand our definition of symmetry to also include $f : \mathbb{R}^4 \to \mathbb{R}^4$ ($\hat{\varphi} = \varphi \circ f$) s.t.

$$\mathcal{L}(x, \hat{\varphi}, \partial_x \hat{\varphi}) = \mathcal{L}(x, \varphi, \partial_x \varphi) + \partial_\alpha V^\alpha.$$

e.g. Consider time translation:

$$\varphi(t, x) \longmapsto \varphi(t + \lambda, x)$$

Then $\mathcal{L} = -\frac{1}{2}\left(g^{\alpha\beta} \overbrace{\varphi_{,\alpha} \varphi_{,\beta}} + m^2 \varphi^2\right)$ means partial derivatives wrt α & β

$g^{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$

Then $\delta\mathcal{L} = -\left(g^{\alpha\beta} \varphi_{,\alpha} \dot{\varphi}_{,\beta} + m^2 \varphi \dot{\varphi}\right)$

$$= \partial_t \mathcal{L}$$

$$= \partial_\alpha (\delta^\alpha_t \mathcal{L}).$$ ← Divergence term

This means there is no preferred instant of time in KG theory

Noether's Thm

Let $\mathcal{L} = \mathcal{L}(x, \varphi, \partial\varphi)$. If $\delta\varphi$ is an infinitesimal variational symmetry, then $\delta\mathcal{L} = 0$.

But also, at any pt in the space of fields

$$\delta\mathcal{L} = \mathcal{E}(\mathcal{L})\,\delta\varphi + D_\alpha V^\alpha$$

where $\mathcal{E}(\mathcal{L}) = \dfrac{\partial\mathcal{L}}{\partial\varphi} - D_\alpha\left(\dfrac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\right)$

& $V^\alpha = \dfrac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\,\delta\varphi$

holds for any field variation

So if $\delta\mathcal{L} = 0$, then $D_\alpha V^\alpha = -\mathcal{E}(\mathcal{L})\,\delta\varphi$

This is exactly the type of identity needed for defining a conserved current V^α: if φ satisfies the KG eqn, then $\mathcal{E}(\mathcal{L}) = 0$ & so

$$\underline{D_\alpha V^\alpha = 0.}$$

Or if $\delta\mathcal{L} = D_\alpha W^\alpha$, then

$$D_\alpha (V^\alpha - W^\alpha) = -\mathcal{E}(\mathcal{L})\,\delta\varphi$$

& ~~the~~ conserved current is $V^\alpha - W^\alpha$.

Noether's First Thm:

If $\delta\varphi(x,\varphi,\partial\varphi,\dots)$ is a divergence symmetry of $\mathcal{L}(x,\varphi,\partial\varphi)$, i.e.

$$\delta\mathcal{L} = D_\alpha W^\alpha,$$

then there exists a conserved current given by

$$j^\alpha = \frac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\,\delta\varphi - W^\alpha.$$

Application: the time translation symmetry gives conserved current defining conservation of energy

Recall the time translation is a divergence symmetry:

$$\delta\varphi = \dot\varphi \Rightarrow \delta\mathcal{L} = D_\alpha(\delta^\alpha_t\mathcal{L})$$

Recognizing that $\delta\varphi = \dot{\varphi}$, $W^\alpha = \delta^\alpha_t \mathcal{L}$, ;

$$\frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}} = -g^{\alpha\beta}\varphi_{,\beta},$$

we have: $j^\alpha = -g^{\alpha\beta}\varphi_{,\beta}\varphi_{,t} - \delta^\alpha_t \mathcal{L} = -T^\alpha_t$

This j^α is a conserved current. We did the calculation $D_\alpha j^\alpha = 0$ earlier when discussing conservation of energy.

We sometimes say the conserved current for energy is the Noether current associated to time translation symmetry

This was predictable since $\mathcal{L}$, in the jet space, has no dependence on t, only on φ, $\partial\varphi$.

Space translations & Conservation of Momentum

Let $\hat{n}$ be a const unit vect field on $\mathbb{R}^3$. The space translation is given by $\varphi(t,\vec{x}) \longmapsto \varphi_\lambda(t,\vec{x}) \doteq \varphi(t,\vec{x}+\lambda\hat{n})$

$\vec{x} = (x,y,z)$

Then
$$\delta\varphi \doteq \left.\frac{d\varphi_\lambda}{d\lambda}\right|_{\lambda=0} = \partial_x\varphi_\lambda(t,\vec{x})\cdot\hat{n}_x + \partial_y\varphi_\lambda(t,\vec{x})\cdot\hat{n}_y + \partial_z\varphi_\lambda(t,\vec{x})\cdot\hat{n}_z$$

$$= \hat{n}\cdot\nabla\varphi_\lambda(t,\vec{x})\Big|_{\lambda=0}$$

$$= \hat{n}\cdot\nabla\varphi.$$

$$= n_i\cdot\varphi_{,i}$$

To check it is a symmetry, compute

$$\delta\mathcal{L} = \delta\left(\tfrac{1}{2}(\dot{\varphi}^2 - |\nabla\varphi|^2 - m^2\varphi^2)\right)$$

$$= \dot{\varphi}\,\delta\dot{\varphi} - \nabla\varphi\cdot\nabla\delta\varphi - m^2\varphi\cdot\delta\varphi$$

$$= \dot{\varphi}(\hat{n}\cdot\nabla\dot{\varphi}) - \nabla\varphi\cdot\nabla(\hat{n}\cdot\nabla\varphi) - m^2\varphi(\hat{n}\cdot\nabla\varphi)$$

$\hat{n}$ is const

$$= \hat{n}\left(\dot{\varphi}\cdot\nabla\dot{\varphi} - \nabla\varphi\cdot\nabla^2\varphi - m^2\varphi\cdot\nabla\varphi\right)$$

On the other hand:

$$\nabla \mathcal{L} = \dot{\varphi}\, \nabla \dot{\varphi} - \nabla\varphi \cdot \nabla^2 \varphi - m^2 \varphi \nabla \varphi$$

So $\delta \mathcal{L} = \hat{n} \cdot \nabla \mathcal{L} = \nabla(\hat{n} \cdot \mathcal{L})$

$= D_\alpha (W^\alpha)$ where

total derivative $\quad W^\alpha = (0, n^i \mathcal{L})$

So space translation = divergence symmetry.

Then, $j^\alpha = (\rho, j^i) = \frac{\partial \mathcal{L}}{\partial \varphi_{,\alpha}} \delta\varphi - W^\alpha$ by Noether's Thm

So $\rho = \dot{\varphi}\, \delta\varphi - 0 = \dot{\varphi}\, \hat{n} \cdot \nabla \varphi$.

Since $|\nabla\varphi|^2 = \varphi_{,1}^2 + \varphi_{,2}^2 + \varphi_{,3}^2$, then

$$j^i = -\varphi_{,i}\, \hat{n} \cdot \nabla\varphi - \frac{n^i}{2} \mathcal{L}$$

$\rho = \dot{\varphi}\,\hat{n}\cdot\nabla\varphi$, $j^i = -\varphi_i\,\hat{n}\cdot\nabla\varphi + \frac{1}{2}n^i\left(|\nabla\varphi|^2 - \dot{\varphi}^2 + m^2\varphi^2\right)$

Let's verify $\frac{d\rho}{dt} + \nabla\cdot\vec{j} = 0$.

$$\frac{d\rho}{dt} = \ddot{\varphi}\,\hat{n}\cdot\nabla\varphi + \dot{\varphi}\,\hat{n}\cdot\nabla\dot{\varphi}$$

cancel

Once summed over i, these cancel.

$$\partial_i j^i = -\varphi_{ii}\,\hat{n}\cdot\nabla\varphi - \varphi_i\,\hat{n}\cdot\nabla\varphi_i + n^i\left(\nabla\varphi_i\cdot\nabla\varphi - \dot{\varphi}\,\dot{\varphi}_i + m^2\varphi\,\varphi_i\right)$$

$$\Rightarrow \nabla\cdot\vec{j} = -\nabla^2\varphi\,\hat{n}\cdot\nabla\varphi - \dot{\varphi}\,\hat{n}\cdot\nabla\dot{\varphi} + m^2\varphi\,\hat{n}\cdot\nabla\varphi$$

$$\frac{d\rho}{dt} + \nabla\cdot\vec{j} = \ddot{\varphi}\,\hat{n}\cdot\nabla\varphi - \nabla^2\varphi\,\hat{n}\cdot\nabla\varphi + m^2\varphi\,\hat{n}\cdot\nabla\varphi$$

$$= \underbrace{-(\Box - m^2)\varphi}_{=\,0 \text{ since } \varphi \text{ is a KG field.}}\cdot\hat{n}\cdot\nabla\varphi = 0.$$

Since $\hat{n}$ is arbitrary, we actually have 3 indep conservation laws for 3 linearly indep choices of $\hat{n}$.

Angular momentum.

Q: What symmetries conserve angular momentum?

A: Lorentz symmetries.

def. A Lorentz symmetry of $\mathbb{R}^4$ is a linear transf

$$x^\alpha \longmapsto S^\alpha_\beta x^\beta$$

s.t. the quadratic form $g_{\alpha\beta} x^\alpha x^\beta = -t^2 + x^2 + y^2 + z^2$ is invariant.

So $S^\alpha_\gamma S^\beta_\delta g_{\alpha\beta} = g_{\gamma\delta}$ $(*)$

Let $S(\lambda)$ be a 1-param family of Lorentz symm.

s.t. $S^\alpha_\beta(0) = \delta^\alpha_\beta$, $\omega^\alpha_\beta \doteq \left(\frac{\partial S^\alpha_\beta}{\partial \lambda}\right)\Big|_{\lambda=0}$

Differentiating $(*)$ wrt λ we get: $\omega^\alpha_\gamma g_{\alpha\delta} + \omega^\beta_\delta g_{\gamma\beta} = 0$

Define $\omega_{\alpha\beta} = g_{\beta\gamma}\,\omega^{\gamma}_{\alpha}$. Then Lorentz transf are generated by ω iff $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$.

So the Lie algebra consists of 4×4 matrices X s.t. $gXg = -X^{T}$ $\left(g = \mathrm{diag}(-1,1,1,1)\right)$.

Then $\delta\varphi = (\omega^{\alpha}_{\beta}\, x^{\beta})\,\varphi_{,\alpha}$

$$\delta\mathcal{L} = D_{\alpha}(\omega^{\alpha}_{\beta}\, x^{\beta}\mathcal{L})$$

& $j^{\alpha} = \omega_{\gamma\delta}\underbrace{M^{\alpha(\gamma)(\delta)}}$

conserved currents associated to relativistic angular momentum

Claim:

All continuous isometries of flat spacetime are contained in the Poincaré group = diffeomorphisms generated by Lorentz transforms & spacetime translations.

Internal Symmetries

These are symmetries on the space of fields, not on spacetime. E.g. $\varphi \longmapsto -\varphi$.

Here, unless $m=0$, there are no interesting continuous internal symmetries.

eg $m=0$, $\varphi_\lambda = \varphi + \lambda$

Then $\delta\varphi = 1$ & $\delta\mathcal{L} = \dot{\varphi}\,\delta\dot{\varphi} - \nabla\varphi \cdot \underset{\overset{\shortparallel}{0}}{\nabla\delta\varphi}$

$$\delta\dot{\varphi} = \left.\frac{d\dot{\varphi}_\lambda}{d\lambda}\right|_{\lambda=0} = \underset{D_\alpha W^\alpha}{\overset{\shortparallel}{0}}$$

So $j^\alpha = \dfrac{\partial\mathcal{L}}{\partial\varphi_{,\alpha}}\,\delta\varphi - \underbrace{W^\alpha}_{\text{const?}}$

Seems $\rho = \dot{\varphi}$ $\Rightarrow$ $\dfrac{d\rho}{dt} + \nabla\cdot\vec{j} = -\Box\varphi = 0$

$j^i = -\varphi_i$

since $m=0$ &

so $\Box\varphi = 0$.

Charged KG Field & its internal symmetries.

A charged KG field is a $\mathbb{C}$-valued f^n

$$\varphi : M \longrightarrow \mathbb{C} \qquad g_{\alpha\beta} = g^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$$

The Lagrangian is now $\mathcal{L} = -g^{\alpha\beta}\left(\varphi_{,\alpha}\varphi^*_{,\beta} + m^2|\varphi|^2\right)$

We can write this Lagrangian as the sum of two $\mathbb{R}$-valued Lagrangians w/ masses $m_1 = m_2 = m$ & fields φ_1, φ_2.

Then the charged field can be written as

$$\varphi = \frac{1}{\sqrt{2}}\left(\varphi_1 + i\varphi_2\right).$$

So really, we can work in real & imaginary parts if we want: φ & φ^*

The Euler-Lagrange eqns are now:

$$\mathcal{E}_\varphi(\mathcal{L}) = (\Box - m^2)\varphi^* = 0$$

$$\mathcal{E}_{\varphi^*}(\mathcal{L}) = (\Box - m^2)\varphi = 0$$

Working on $\mathbb{C}$ gives new deg of freedom which allows us to introduce conserved electric charge. In QFT, we get anti-particles.

The charged Lagrangian now admits the internal continuous symmetry $\varphi_\alpha = e^{i\alpha}\varphi$, $\varphi^*_\alpha = e^{-i\alpha}\varphi^*$.

These are sometimes called phase transformations or rigid $U(1)$ transformations.

See p. 66-7 for the conserved current - the outcome is

$$j^\alpha = -ig^{\alpha\beta}(\varphi^* \varphi_{,\beta} - \varphi \varphi^*_{,\beta})$$

Let $V \subset \mathbb{R}^3$ at a fixed time. Then the total $U(1)$ charge

is $Q = i\int_V (\varphi^* \dot{\varphi} - \varphi \dot{\varphi}^*)\, d^3x$

Use this for modelling electric charge in electrodynamics

Can also use it to model charge which interacts w/ neutral currents in electroweak theory.

Of course, we can do all this for general groups w/ representations on vect space V.

Consider $\varphi : M \to V$ w/ internal symmetries given by $r: G \to \mathrm{Aut}(V)$.

Or even, let V be a vector bundle & $\varphi : M \to V$ a section

Let $G = SU(2)$, $V = \mathbb{C}^2$.

an element $U \in SU(2)$ can be written as

$$U(\theta, n) = \cos\left(\frac{\theta}{2}\right) I + i \sin\left(\frac{\theta}{2}\right) n^i \sigma_i$$

$n = (n^1, n^2, n^3)$, $\sum_{j=1}^{3} (n^j)^2 = 1$ &

Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

If $U(\lambda)$ is any 1-param family in $SU(2)$, w/ $U(0) = I$

& $\varphi_\lambda = U(\lambda)\varphi$, then

$\delta\varphi = i\tau\varphi$, where τ is a Hermitian, traceless 2×2 matrix

defined by $\tau = \frac{1}{i}\left(\frac{dU}{d\lambda}\right)_{\lambda=0}$

So $i\tau \in \mathfrak{su}(2)$.

Of course, τ is a $\mathbb{R}$ linear combo of the σ_i

Hermitian inner prod on $\mathbb{C}^2$: $(\varphi_1, \varphi_2) = \varphi_1^* \varphi_2$

Of course, it is $U(n)$ invariant. So we build a Lag.

$$\mathcal{L} = -\left[g^{\alpha\beta}(\varphi_{,\alpha}, \varphi_{,\beta}) + m^2(\varphi, \varphi)\right]$$

Claim: for symmetry $\delta\varphi = i\tau\delta$, the assoc conserved current is $j^\alpha = i g^{\alpha\beta}(\varphi^*_{,\beta}\tau\varphi - \varphi^* \tau \varphi_{,\beta})$.

There are 3 indep conserved currents here b/c of τ.

The 3 conserved charges assoc w/ SU(2) symm are called isospin

Claim: The converse to what we wrote as Noether's 1st Thm is true:

For each conservation law for a system of Euler-Lagrange eqns, there is a corresponding symmetry of the Lagrangian.

For many types of field theories, we also have 1-1 correspondence b/w conservation laws & symmetries of the Lagrangian.

(including KG field theory)

Let $M = (\mathbb{R}^{2n}, \omega)$ (phase space) w/ $H : \mathbb{R}^{2n} \to \mathbb{R}$ a Hamiltonian. Let $\gamma : [0,1] \to \mathbb{R}^{2n}$ be a classical traj.

Then for any function F, $\{F, H\} = \frac{d}{dt} F(\gamma(t))$.

This is b/c we define $\{F, H\} \doteq -(\nabla F)^T J_0 \nabla H$

$$= (\nabla F)^T \dot{\gamma}$$

$$= \frac{d}{dt}(F \circ \gamma)$$

More generally, since $\{\,,\,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, take another function G; we can think of G as the infinitesimal generator of a transformation:

$$\delta F = \{F, G\}$$

Then, if $\delta H = \{H, G\} = 0$, we call G a symmetry of the Hamiltonian system.

But $\dot{G} = \{G, H\} = -\{H, G\} = 0$

So G is also a conserved quantity along class. traj.

i.e. A f^n on phase space generates a symmetry iff it is conserved.