Classical Field Theory — Charles Torre

A mechanical system is a dynamical system w/ finitely many degrees of freedom.

A field is also a dynamical system but w/ infinitely many degrees of freedom.

Mathematically, a field is a section of a fiber bundle.

Let $\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}$ be a scalar field

$x^\alpha = (t, x, y, z)$

The Klein-Gordon Equation is

$$\Box \varphi - m^2 \varphi = 0$$

where $\Box = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$

a wave operator called the d'Alembertian

When $m=0$, we get the wave eqn:

$$\Delta \varphi = \partial_t^2 \varphi$$

Laplacian

The KG eqn came about as an attempt to give a relativistic Schrödinger eqn but this did not work; afterall, Schrödinger is mechanical, i.e. finitely many deg of freedom

But KG is sort of a classical limit of a quantum field in relativistic settings

How to solve KG? Suppose φ is well behaved, such as

$$\varphi_t \in L^2(\mathbb{R}^3) \quad \forall t.$$

Taking a Fourier expansion:

$$\varphi(t,r) = \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbb{R}^3} \hat{\varphi}_k(t)\, e^{ik\cdot r}\, d^3k$$

We have $\hat{\varphi}_{-k} = \hat{\varphi}_k^*$ since φ is real valued.

If φ satisfies $\Box\varphi = m^2\varphi$, then

$$\int -\partial_t^2\left(\hat{\varphi}_k(t)\right)e^{ik\cdot r}\,d^3r$$

$$+\int \partial_x^2\left(\hat{\varphi}_k e^{ik\cdot r}\right) + \partial_y^2\left(\hat{\varphi}_k e^{ik\cdot r}\right) + \underbrace{\partial_z^2\left(\hat{\varphi}_k e^{ik\cdot r}\right)}_{\shortparallel}\,d^3r$$

$$= m^2\int \hat{\varphi}_k \cdot e^{ik\cdot r}\,d^3r$$

$\hat{\varphi}_k \cdot \partial_z^2\left(e^{ik\cdot r}\right)$

$= -k_z^2\,\hat{\varphi}_k\,e^{ikr}$

$\Rightarrow$ In the integrands:

$$\left(-\ddot{\hat{\varphi}}_k - \vec{k}\cdot\vec{k}\,\hat{\varphi}_k\right)e^{ik\cdot r} = m^2\hat{\varphi}\,e^{ik\cdot r}$$

dot product

$$\Rightarrow \ddot{\hat{\varphi}}_k + (k^2 + m^2)\varphi_k = 0.$$

norm square

This is easy to solve; it's just a 2nd order ODE of a simple form.

$$\hat{\varphi}_k(t) = a_k e^{i\omega_k t} + b_k e^{-i\omega_k t}$$

$$\omega_k = \sqrt{k^2 + m^2}$$

Since $\hat{\varphi}_{-k} = \hat{\varphi}_k^*$, then $b_{-k} = a_k^*$.

$$\therefore \varphi(k) = \left(\frac{1}{2\pi}\right)^{3/2} \int_{\mathbb{R}^3} \left(a_k e^{i(k\cdot r - \omega_k t)} + a_k^* e^{-i(k\cdot r - \omega_k t)} \right) d^3 r$$

KG field

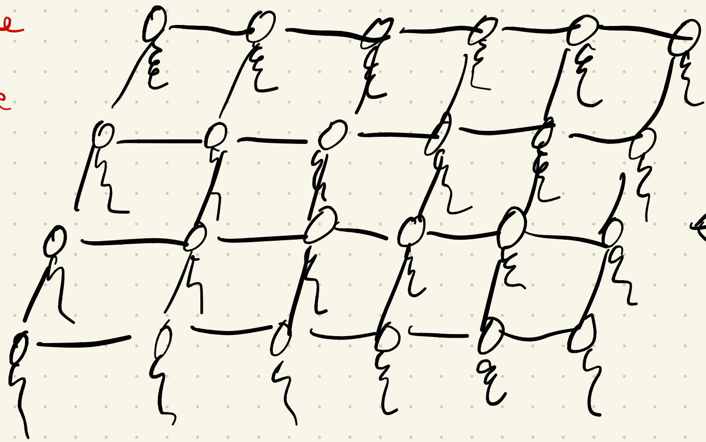
Note: The solution φ to KG is essentially an infinite collection of uncoupled harmonic oscillators for each $k \in \mathbb{R}^3$

So there are infinite deg of freedom

Also, the KG field is often called the free or non-interacting field b/c of the uncoupled nature of the oscillators

So picture an array of springs as a visual of a Field φ satisfying KG

Discrete picture



nudge it here to "excite" the array. Then a wave will propagate outwards.

This excitation which leads to a wave is what we call a particle. Particles are waves in QM.

Let $L = \frac{1}{2}\int_{\mathbb{R}^3} \underbrace{\left(\dot{\varphi}^2 - |\nabla\varphi|^2 - m^2\varphi^2\right)}_{= \mathcal{L} \text{ (Lagrangian density)}} d^3\cancel{x}$ (Lagrangian on space)

include time now } $S[\varphi] = \int_{t_1}^{t_2} L\, dt$, (action), let $R = [t_1, t_2] \times \mathbb{R}^3$

Variation of S.

Let $\lambda \in \mathbb{R}$ be a parameter } φ_λ a 1-parameter family of fields. w/ $\varphi_0 = \varphi$.

$$\delta S = \left.\frac{d\, S[\varphi_\lambda]}{d\lambda}\right|_{\lambda=0}. \quad \text{Also } \delta\varphi \doteq \left.\frac{d\varphi_\lambda}{d\lambda}\right|_{\lambda=0}.$$

Then
$$\delta S = \int_R \left(\dot{\varphi}\,\delta\dot{\varphi} - \nabla\varphi\cdot\nabla\delta\varphi - m^2\varphi\,\delta\varphi\right) d^4x.$$

Note: $\nabla\cdot(\nabla\varphi\,\delta\varphi) = \nabla^2\varphi\,\delta\varphi + \nabla\varphi\cdot\nabla\delta\varphi.$

Using integration by parts, note:

$$\int \dot{\varphi}\, \delta\dot{\varphi} = \dot{\varphi}\, \delta\varphi - \int \ddot{\varphi}\, \delta\varphi .$$

$$\therefore \quad \delta S = \int_{R} \left(-\ddot{\varphi} + \nabla^2\varphi - m^2\varphi\right)\delta\varphi \, d^4x + \left[\int_{\mathbb{R}^3} \dot{\varphi}\, \delta\varphi\right]_{t_1}^{t_2}$$

$$\underbrace{- \int_{R} \nabla\cdot\left(\nabla\varphi\, \delta\varphi\right) d^4x.}_{= -\int_{t_1}^{t_2} dt \int_{r\to\infty} n\cdot\nabla\varphi\, \delta\varphi \, d^2A}$$

(Divergence Theorem)

If φ has cpt support or $\to 0$ faster than $\frac{1}{r^2}$, then the last term (boundary term) vanishes.

Other boundary conditions, such as the endpoints of φ_t are fixed will make $\delta\varphi|_{t_1} = \delta\varphi|_{t_2} = 0$.

This will force the middle term to vanish.

$\therefore$ w/ all these conditions,

$$\delta S = 0 \text{ when } \int_R \left(-\ddot{\varphi} + \nabla^2\varphi - m^2\varphi\right)\delta\varphi \, d^4x = 0$$

$$\Rightarrow \Box\varphi - m^2\varphi = 0 \text{ everywhere in } R.$$

So S is the correct action for KG eqn

Another view is via Euler-Lagrange eqn.

$$\mathcal{L} = \frac{1}{2}\left(\dot{\varphi}^2 - |\nabla\varphi|^2 - m^2\varphi^2\right)$$

Treating $x, \varphi, \varphi_\alpha$ as formal variables (so $\mathcal{L}$ is in the 1st Jet space)

define $$E(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial \varphi} - D_\alpha \frac{\partial \mathcal{L}}{\partial \varphi_\alpha}$$

total derivative

Then $\dfrac{\delta S}{\delta \varphi} = \mathcal{E}(\mathcal{L})\Big|_{\varphi = \varphi(x)} = \Box\varphi - m^2\varphi.$

$\in J^2$ (2^{nd} jet space)

In fact, if $\hat{\mathcal{L}} = \frac{1}{2}\varphi(\Box - m^2\varphi)$, one can show

$$\mathcal{E}(\mathcal{L}) = \mathcal{E}(\hat{\mathcal{L}}) \quad \& \quad \mathcal{L} = \hat{\mathcal{L}} + \text{divergence term}$$

sum of all the 1^{st} partial derivatives of something.

Some Generalizations of KG.

Let $\mathcal{L} = \frac{1}{2}(\dot{\varphi}^2 - |\nabla\varphi|^2 - m^2\varphi^2) - j\varphi$

where $j : \mathbb{R}^4 \to \mathbb{R}$ is the "source" (in electromagnetism, j could be electric charge or current)

Then $\mathcal{E}(L) = (\Box - m^2)\varphi - j.$

In QFT, the presence of a source leads to particle creation/annihilation via transfer of energy-momentum b/w the field & its source.

The KG eqns are linear ; so the solutions are non-interacting.

If we modify KG to be non-linear, we introduce self-interaction.

potential

E.g. $\mathcal{L} = \frac{1}{2}(\dot{\varphi}^2 - |\nabla\varphi|^2 - m^2\varphi^2) - V(\varphi)$

$E(L) = (\Box - m^2)\varphi - V'(\varphi) = 0.$

So long as V isn't quadratic, this becomes non-linear.

One potential of interest: $V(\varphi) = -\frac{1}{2}a^2\varphi^2 + \frac{1}{4}b^2\varphi^4$.

If V is quad, then $V'(\varphi) = a\varphi - b$

; so $(\Box - (m^2+a)\varphi) = b$

Seems like, you just increase the mass... what does this mean, physically?

Coordinate free description: Let (M,g) be a Lorentzian mfd.

Then $\mathcal{L} = -\frac{1}{2}\left(g^{-1}(d\varphi, d\varphi) + m^2\varphi^2\right)\varepsilon(g)$

volume form of g.

is the Lagrangian density.

This is called minimally coupled

Let ξ be a parameter & $R(g)$ = scalar curvature. Then

$$\mathcal{L} = -\frac{1}{2}\left[g^{-1}(d\varphi, d\varphi) + (m^2 + \xi R(g))\varphi^2\right]\varepsilon(g)$$ gives

Curvature coupled KG theory.

These theories are not "diffeomorphism invariant" or "generally covariant"; i.e. $f : M \xrightarrow{\text{diffeo}} M$ & $\hat{g} = f^* g$, then

$$\hat{\mathcal{L}} = -\frac{1}{2}\left(\hat{g}^{-1}(d\varphi, d\varphi) + m^2\varphi^2\right)\varepsilon(\hat{g})$$ is a new

Lagrangian density in general unless f is a symmetry.

If we allow g to vary, we get 11 coupled non-linear field eqns instead of 1 linear field eqn.

Conservation laws are fundamental & give info about complicated dynamics. Also, conservation laws are related to symmetries by Noether's theorem.

def: Let $j^\alpha = j^\alpha(x, \varphi, \partial\varphi, \ldots, \partial^k\varphi) \in \mathcal{J}^k$ be a vector field constructed as a local f^n. j^α is a conserved current or defines a conservation law if the divergence of $j^\alpha = 0$ when φ satisfies the field eqn.

e.g. $D_\alpha j^\alpha = 0$ when $(\Box - m^2)\varphi = 0$.

Explicitly, if $\varphi(x)$ is a solution, take

$$j^\alpha(x) \doteq j^\alpha\left(x, \varphi(x), \frac{\partial\varphi(x)}{\partial x}, \dots, \frac{\partial^k\varphi(x)}{\partial x^k}\right) \quad \text{s.t.}$$

$$\frac{\partial}{\partial x^\alpha} j^\alpha = 0.$$

If $j^\alpha = (j^0, j^1, j^2, j^3)$, call $\rho \doteq j^0$ the density

$\vec{j} = (j^1, j^2, j^3)$ the current density

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0.$$

The point of writing this is:

Let $Q_V(t) = \int_V \rho(t, \vec{x})\, d^3x$ be the total charge in region V.

Then
$$\frac{d}{dt} Q_V(t) = -\int_V \nabla \cdot \vec{j} = -\underbrace{\int_{\partial V} \vec{j}\, d^2S}_{\text{net flux}}$$

We say Q_V is conserved since we can see how it changes over time by purely in terms of the net flux, a fixed value. So there's no creation nor destruction of charge; it just moves around.

If we place boundary conditions, such as $V = \mathbb{R}^3$; the field vanishes rapidly enough at ∞, then $\frac{d}{dt} Q_V(t) = 0$; so the total charge is constant

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Conservation of Energy

Let $j^0 = \frac{1}{2}(\dot{\varphi}^2 + |\nabla\varphi|^2 + m^2\varphi^2)$

$j^i = -\dot{\varphi}(\nabla\varphi)_i$

$$\left.\begin{aligned} \partial_0 j^0 &= \dot{\varphi}\ddot{\varphi} + \nabla\varphi \cdot \nabla\dot{\varphi} + m^2\varphi \cdot \dot{\varphi} \\ \underbrace{\partial_i j^i}_{\text{sum the Einstein notation}} &= -\nabla\dot{\varphi} \cdot \nabla\varphi - \dot{\varphi}\nabla^2\varphi \end{aligned}\right] \Rightarrow \partial_\alpha j^\alpha = -\dot{\varphi}(\Box\varphi - m^2\varphi)$$

So if φ is a solution to KG, $\partial_\alpha j^\alpha = 0$.

Let $E_V = \frac{1}{2}\int_V \left(\dot\varphi^2 + |\nabla\varphi|^2 + m^2\varphi^2\right) d^3x$

total energy

$= T + U$

$T = \frac{1}{2}\int_V \dot\varphi^2 \, d^3x$ kinetic energy

$U = \frac{1}{2}\int_V \left(|\nabla\varphi|^2 + m^2\varphi^2\right) d^3x$ potential energy

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Then $\rho = j^0$ $\vec{j} = (j^1, j^2, j^3)$

$$\frac{d}{dt} E_V = \int \partial_t \rho = -\int \nabla\cdot\vec{j} = -\int_V \overbrace{\left(\nabla\varphi\cdot\nabla\dot\varphi + \dot\varphi\,\nabla^2\varphi\right)}^{\nabla\cdot(\dot\varphi\,\nabla\varphi)} d^3x$$

$$= -\int_{\partial V} \dot\varphi \, \nabla_n\varphi \, dS.$$

If $\dot\varphi = 0$ or the normal component of $\nabla\varphi$ to ∂V vanishes,

then $\frac{d}{dt} E_V = 0$. $\{$ Energy is conserved.