# Outline of Ch. 9: Space of Trajectories and Gluing

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We give an outline of ch. 9 of *Morse Theory and Floer Homology*.

The goal of Floer theory is to create the Floer chain complex and define a homology theory. Much like in Morse theory, for a nondegenerate Hamiltonian $H$ on our compact symplectic manifold $(M, \omega)$, we consider the space of trajectories of finite energy $\mathcal{M}$. A theorem of ch. 6 shows that $\mathcal{M}$ is the union of $\mathcal{M}(x, y)$: finite-energy trajectories connecting periodic orbits. Therefore, we define $C_k(H)$ to be the $\mathbb{Z}_2$ vector space generated by periodic orbits of Maslov index $k$. The nondegeneracy condition guarantees finitely many critical points of $\mathcal{A}_H$. The differential $\partial : C_k(H) \to C_{k-1}(H)$ is defined by

$$\partial x = \sum_{y \in C_{k-1}} n(x, y) y.$$

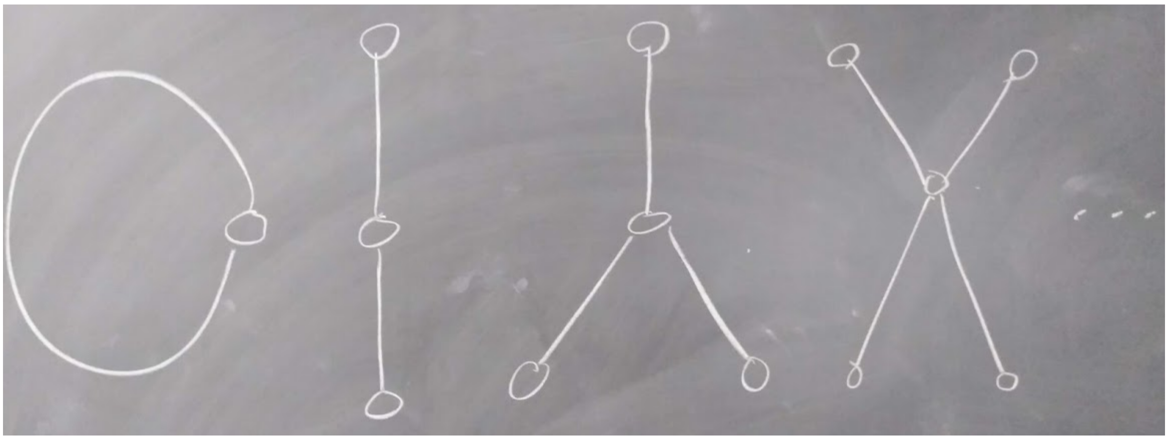
Here $n(x, y)$ is the number of trajectories connecting $x$ and $y$, mod 2. Again, these trajectories are the finite energy contractible solutions of the Floer equation connecting $x$ and $y$.

For this differential to make sense, we need to show that counting trajectories makes sense: if $\mu(x) = \mu(y) + 1$, then $\mathcal{L}(x, y) = \mathcal{M}(x, y)/\mathbb{R}$ should be a 0-dim manifold with finitely many points. When $\mu(x) = \mu(y) + 2$, then $\mathcal{L}(x, y)$ is a 1-manifold and if we compactify it, it will be a 1-manifold with an even number of boundary points.

It is not hard to show that when the indices differ by two, that $\mathcal{L}(x, y)$ is a 1-manifold and that $\overline{\mathcal{L}}(x, y)$, is compact. What is hard is showing $\overline{\mathcal{L}}(x, y)$ is a 1-manifold with boundary. The majority of ch. 9 is dedicated to this. For general indices, $\overline{\mathcal{L}}(x, y)$ is a manifold with *corners*.

## 0 A General Outline

Let me reiterate the goal. We want to define the differential $\partial$ and in the case that the indices differ by 2, the moduli spaces we're looking at are open 1-manifolds. It is feasible that the 1-manifold looks like three open intervals arranged so that if we added in a single point $(\hat{u}, \hat{v})$, then we connect the three intervals into a sort of triangular looking thing. Compactifying such a thing adds in four points but this would no longer be a 1-manifold. Moreover, it would have three boundary points and so $\partial^2 \neq 0$. Thus, we want to rule out such situations.



We want to rule out such pre-compact moduli spaces

To rule this out, we need to show that there is only one way to approach $(\hat{u}, \hat{v})$ in a sequence as opposed to three ways in the example above or any other number of ways greater than one. **This is what we mean by "uniqueness of the gluing."** There is exactly one approach. Note that in the picture above, there's a circle with a point removed. Certainly it is possible that the compactified moduli space has circles in it but the uniqueness result implies that prior to compactifying, the moduli space already had a circle. We wouldn't be adding in a point to form a circle.

To show uniqueness, we first show that if we have a sequence $\ell_n$ very close to $(\hat{u}, \hat{v})$, they start to take on a very particular form up to a parameter $\rho$: they coincide with $u(s+\rho, t)$ and $v(s-\rho, t)$ when $s \leq -1$ and $s \geq 1$, resp. On $[-1, 1]$, they essentially look like some smooth patching together of exponentials $\exp Y$ and $\exp Z$ for some vectors $Y, Z$. This smooth patching is not from arbitrary vectors $Y$ and $Z$; the vectors have some particular properties related to the trajectories. This is quite a powerful statement: **all trajectories converging to a broken trajectory eventually have a very specific form.**

The next step is to show that elements of this very particular form, called **pregluings** $w_\rho$ converge in a unique fashion; we can't approach $(\hat{u}, \hat{v})$ from more than one direction. Another way to put it is that eventually, the $\ell_n$ are in the image of an embedding $\hat{\psi}$.

There is a caveat however. Above, I said that near $(\hat{u}, \hat{v})$, things start to look like $u(s+\rho, t)$ and $v(s-\rho, t)$. However, they can have a somewhat more general form. We could replace $\rho$ with $\nu(\rho)$ where $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ is a function increasing to $+\infty$. In such a case, we would get an embedding $\hat{\psi}_\nu$ with all the same properties as if we had taken $\nu = \mathrm{id}$. We need to show that ultimately, this choice of $\nu$ doesn't make a difference.

# 1 The Space of Trajectories

Let $\mathcal{L}(x, y) = \mathcal{M}(x, y)/\mathbb{R}$ be give the quotient topology. Here is a proposition:

**Proposition 1.1.** *Let $x$ and $y$ be two distinct critical points of $\mathcal{A}_H$ and let $u_n \in \mathcal{M}(x, y)$, $s_n, \sigma_n \in \mathbb{R}$. Suppose further that:*

$$\lim u_n(s_n + s, \cdot) = u \in \mathcal{M}(x, z) \quad \textit{and} \quad \lim u_n(\sigma_n + s, \cdot) = v \in \mathcal{M}(x, w)$$

*for two critical points $z, w$ distinct from $x$. Then $z = w$ and $u$ and $v$ coincide up to action by $\mathbb{R}$. In other words, there exists $s^*$ such that $u(s^* + s, t) = v(s, t)$.*

This proposition says that however we translate the $u_n$, the limit of $u_n \cdot s_n$ and $u_n \cdot \sigma_n$ coincide up to $\mathbb{R}$ action. So they both converge to some trajectory between $x$ and $z = w$. This gives uniqueness of limits and thus, $\mathcal{L}(x, y)$ is **Hausdorff.**

The next theorem will let us define a compactification of $\mathcal{L}(x, y)$.

**Theorem 1.2.** *Let $(u_n)$ be a sequence of elements of $\mathcal{M}(x, y)$. There exists:*

- *A subsequence of $(u_n)$; continue calling it $(u_n)$*
- *critical points $x_0 = x, x_1, ..., x_l, x_{l+1} = y$*
- *Sequences $(s_n^k) \in \mathbb{R}$ for $0 \leq k \leq l$.*
- *elements $u^k \in \mathcal{M}(x_k, x_{k+1})$*

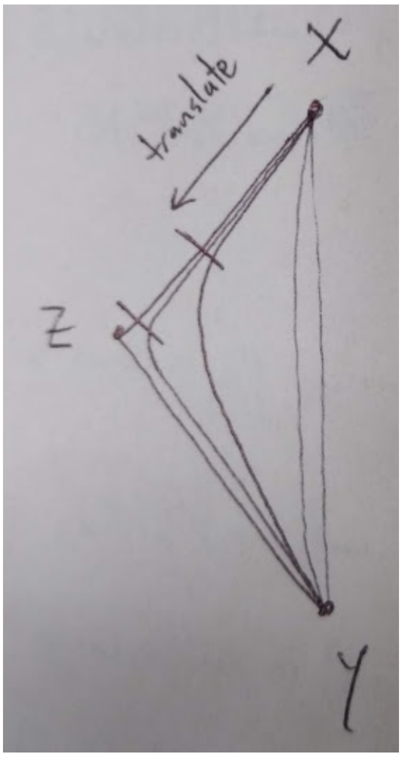
*Such that for every $k = 0, ..., l$,*

$$\lim_{n\to\infty} u_n \cdot s_n^k = u^k.$$

This theorem says that by translating a subsequence $(u_n)$ with carefully chosen values $s_n^k$, the limit goes to some $u^k \in \mathcal{M}(x_k, x_{k+1})$. This may seem weird because given some $u \in \mathcal{M}$, translation by $s_0$ gives another solution $u \cdot s_0$ which has the same image as $u$ in $M$. However, the topology of $\mathcal{M}$ is that of $C^\infty_{\text{loc}}$. Recall the following example which is a modification of the sliding block example for $L^1_{\text{loc}}$:

**Example 1.3.** [Sliding Block] Fix $\tau \in \mathbb{R}$. Let $\beta_{[\tau,\infty)}$ be a smooth bump function which is 1 on $[\tau, \infty)$ and 0 on $(-\infty, \tau - 1]$. Then translation by a positive sequence $t_k \to \infty$ pushes the positive part of $\beta$ to $+\infty$. So then $\beta_{t_k} = \beta \cdot t_k$ converges to 0. But if $t_k \to -\infty$, then $\beta_{t_k}$ converges to the constant function 1. And we can let $t_k$ converge to some constant $c$ to have the bump functions converge a bump starting at $\tau + c$.

This example shows that in $C^\infty_{\text{loc}}$, translating differently does indeed give a different limit. Thus, in our theorem above, depending on how we translate via $s_n^k$, we get the $u_n$ converging to different $u^k$. Large "chunks" of $u_n$ are pushed to $\pm\infty$ and thus the $u^k$ themselves converge to $x_k(t)$ and $x_{k+1}(t)$ as $s \to \pm\infty$.




Translation of Floer Solutions

Part of the proof of this theorem is analogous to the Morse theory result. It also involves covering the critical points with balls of radius $\epsilon$ where $\epsilon$ is small enough that all the balls are disjoint. One then uses these balls to detect and mark when a trajectory enters or exits; this information allows one to say whether a trajectory converges to a critical point or not.

# 2 Gluing

The main goal of this section is to prove the gluing theorem:

**Theorem 2.1** (Gluing)**.** *Let $x, y, z$ be critical points with consecutive indices ($x$ has the highest). Let $(u, v) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z)$ represent trajectories $(\hat{u}, \hat{v}) \in \mathcal{L}(x, y) \times \mathcal{L}(y, z)$. We then have:*

- *a differentiable map $\psi : [\rho_0, \infty) \to \mathcal{M}(x, z)$ for some $\rho > 0$, such that $\hat{\psi} = \pi \circ \psi : [\rho_0, \infty) \to \mathcal{L}(x, z)$ is an embedding, satisfying*

$$\lim_{p \to \infty} \hat{\psi}(\rho) = (\hat{u}, \hat{v}) \in \overline{\mathcal{L}}(x, z).$$

- *Moreover, if $\ell_n \in \mathcal{L}(x, z)$ is a sequence that tends to $(\hat{u}, \hat{v})$, then $\ell_n \in Im\ (\hat{\psi})$ for $n$ sufficiently large.*

First, a note. The $\rho_0$ here is chosen to be large enough. The phrase "$\rho$ sufficiently large" will be equivalent to the phrase "$\rho \geq \rho_0$." The purpose of this theorem is to establish that in the Floer chain complex, $\partial^2 = 0$ because we'll be counting boundary points of $\mathcal{L}(x, z)$ taken mod 2. When the difference in index is 2, we get a 1-manifold with boundary and there is only one way to smooth it. But in general, there is no canonical smoothing of these topological manifolds with corners. The notion of log-smooth is sometimes used to study these spaces.
An outline of the proof:

1. Pre-Gluing: we construct an interpolation $w_\rho$ between $u$ and $v$ which depends on the parameter $\rho$. This $w_\rho$ is an approximate solution in the sense that $\mathcal{F}(w_\rho) = 0$ on $|s| \geq 1$.

2. This approximate solution will be used to construct $\psi$ which we write as $\psi(\rho) = \exp_{w_\rho}(\gamma(\rho))$ for some $\gamma(\rho) \in W^{1,p}(w_\rho^* TM) = T_{w_\rho}\mathcal{P}(x, z)$. We want $\psi(\rho)$ to be a true solution of the Floer equation. The way to obtain this $\gamma(\rho)$ is to use the Newton-Picard method. Recall that if $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function, then we can look for solutions $f(x) = 0$ by letting $x_0$ be a approximate solution and then taking

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

   As $n \to \infty$, we get our solution. However, there is a useful variant of this method; instead of computing $1/f'(x_n)$ each time, we can compute just $1/f'(x_0)$ and let $x_{n+1}$ be defined by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$

   In the same way, Picard generalized Newton's method to maps on Banach manifolds, $F : X \to Y$. We only need to compute $(d\mathcal{F})^{-1}_{w_\rho}$ in order to get a unique solution $\gamma(\rho)$.

   In certain literature, we have a Newton-Picard theorem for families, parametrized over $\rho$. Then as long as the function $F : X \times \mathbb{R} \to Y$ is smooth and the approximate solutions vary smoothly, then the true solutions vary smoothly.

3. Verify the three properties of $\hat{\psi}$ from above.

## 2.1 Pre-Gluing

Let us fix a bump function $\beta^+ : \mathbb{R} \to [0,1]$ which is 1 for $s \geq 1$ and 0 for $s \leq \epsilon$. Let $\beta^-(s) = \beta^+(-s)$.

Our interpolation $w_\rho$ is defined by

$$w_\rho(s,t) = \begin{cases} u(s+\rho,t) & s \leq -1 \\ \exp_{y(t)}\left(\beta^-(s)\exp^{-1}_{y(t)}(u(s+\rho,t))) + (\beta^+(s)\exp^{-1}_{y(t)}(v(s-\rho,t))\right) & s \in [-1,1] \\ v(s-\rho,t) & s \geq 1 \end{cases}$$

Indeed, this is a well-defined interpolation for sufficiently large $\rho$. Outside $[-1,1] \times S^1$, it matches $u$ and $v$. The bump functions do some scaling on $[-1,1]$. In fact, it equals $y(t)$ on $[-\epsilon,\epsilon]$. Here are some other properties:

1. $w_\rho \in C^\infty(x,z)$ and for large $\rho$, it is in $C^\infty_\searrow(x,z)$ due to the exponential decay of ch. 8.9.
2. For $s \leq \rho - 1$, we have $w_\rho(s-\rho,t) = u(s,t)$. In particular, we have convergence in $C^\infty_{\text{loc}}$:

   $$\lim_{\rho\to+\infty} w_\rho(s-\rho,t) = u(s,t)$$

   We have the same analogously for $v$.
3. $w_\rho$ is differentiable in $\rho$.
4. $w_\rho(s,t)$ tends to $y(t)$ in $C^\infty_{\text{loc}}$ when $\rho$ tends to $+\infty$.

We might also view this construction as a "connect sum" $w_\rho = u\#_\rho v$. In this way, we then have a map

$$\#_\rho : C^\infty(x,y) \times C^\infty(y,z) \to C^\infty(x,z)$$

and we may consider its differential:

$$T_{(u,v)}\#_\rho) : T_u\mathcal{P}(x,y) \times T_v\mathcal{P}(y,z) \to T_{w_\rho}\mathcal{P}(x,z)$$

This map is exactly as one expects:

$$Y\#_\rho Z(s,t)) = \begin{cases} Y(s+\rho,t) & s \leq -1 \\ T\exp_{y(t)}\big(\beta^-(s)T_{u(s+\rho,t)}\exp^{-1}_{y(t)}(Y(s+\rho,t)) & \\ \qquad\qquad +\beta^+(s)T_{v(s-\rho,t)}\exp^{-1}_{y(t)}(Z(s-\rho,t))\big) & s \in [-1,1] \\ Z(s-\rho,t) & s \geq 1 \end{cases}$$

Here are some properties of $Y\#_\rho Z \in T_{w_\rho}\mathcal{P}(x,z)$:

1. $Y\#_\rho Z$ is an element of $W^{1,p}$ and thus, is continuous.
2. For $s \in [-\epsilon,\epsilon]$, $Y\#_\rho Z = 0$.
3. $\lim_{\rho\to+\infty} Y\#_\rho Z = 0$ in $C^0_{\text{loc}}$. This convergence is of $C^\infty_{\text{loc}}$ if $Y, Z$ are smooth.

## 2.2 Construction of $\psi$

For this construction, we want to find some $\gamma(\rho)$ such that $\psi(\rho) = \exp_{w_\rho}(\gamma(\rho))$ is a solution of the Floer equation. Indeed, it is enough to verify that $\mathcal{F}(\psi_\rho) = 0$ in the weak sense (satisfies an equation in terms of distributions). Since $\psi_\rho$ is continuous as $\gamma(\rho)$ will be continuous, it is automatically a strong solution of $C^\infty$ class due to elliptic regularity.

Part of the construction also involves taking various trivializations, some unitary and others just orthonormal. These are generally denoted $(Z_i)_{i=1,\dots,2n}$. Details of this are on pp. 316-7.

Let us define $\mathcal{F}_\rho = \mathcal{F} \circ \exp_{w_\rho}$ in the basis $(Z_i)$. It's clear that $\mathcal{F}_\rho(0) = \mathcal{F}(w_\rho)$. It is not a linear map as $\mathcal{F}_\rho(0) \neq 0$. However, it does equal 0 outside $[-1,1] \times S^1$ and it converges to 0 as $\rho \to +\infty$ in $L^p$ and $C^\infty$ topology. Taking this approximate solution (0 for $\mathcal{F}_\rho \iff w_\rho$ for $\mathcal{F}$), we use the Newton-Picard method to find a true solution.

Before doing that, let $L_\rho = (d\mathcal{F}_\rho)_0$. Then, in our trivializations,

$$L_\rho(Y) = \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S_\rho(s,t) Y$$

where $S_\rho : \mathbb{R} \times S^1 \to M_{2n}(\mathbb{R})$ is a map into matrices and converges to all the right matrices. The point is that we can conclude that $L_\rho$ is a Fredholm operator with index 2. This is a problem because only index 0 Fredholm maps are invertible. If we wish to produce something analogous to the $1/f'(x_0)$ in the Newton method, we'll need invertibility.

What we do instead is produce a closed complement $W_\rho^\perp$ of $\ker L_\rho$ such that $L_\rho$ restricted to $W_\rho^\perp$ is invertible; i.e. there is a right inverse. It makes sense that we use such a subspace $W_\rho^\perp$: $\mathcal{M}(x,z)$ is a dim 2 manifold and so we want $\gamma \in \exp^{-1} \mathcal{M}(x,z)$. We can then intersect $\exp^{-1} \mathcal{M}(x,z)$ with our codim 2 subspace $W_\rho^\perp$; the resulting space is 0-dim. We then produce our $\gamma$ for each fixed $\rho$.

**Lemma 2.2** (Newton-Picard Method)**.** *Let $X, Y$ be Banach spaces and let $F : X \to Y$ be a continuous map. We write $F(x) = F(0) + L(x) + N(x)$ where $L(x) = (dF)_0(x)$ and we suppose that there exist a continuous $G : Y \to X$ such that:*

1. $L \circ G = \mathrm{id}$
2. $\|GNx - GNy\| \leq C(\|x\| + \|y\|)\|x - y\|$ *for all* $x, y \in B(0,r)$.
3. $\|GF(0\| \leq \epsilon/2$ *where* $\epsilon = \min(r, 1/5C)$.

*Then there exists a unique $\alpha \in Im(G) \cap B(0,\epsilon)$ such that $F\alpha = 0$. Moreover, $\|\alpha\| \leq 2\|GF(0)\|$.*

Writing $F(x) = F(0) + L(x) + N(x)$ is basically a Taylor expansion where $N$ is some "higher terms." Condition (2) basically asks for $\|GNx - GNy\|$ to be bounded by some quadratic term when $x, y$ are nearby to 0. We want this because $N$ has at least quadratic terms.

Let us define $W_\rho$. Let $L^u, L^v$ be the differentials $(d\mathcal{F})_u, (d\mathcal{F})_v$, respectively. Then

$$W_\rho := \{\alpha \#_\rho \beta : \alpha \in \ker L^u, \beta \in \ker L^v\}$$

and

$$W_\rho^\perp := \left\{ Y \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) : \int_{\mathbb{R} \times S^1} \langle Y, \alpha \#_\rho \beta \rangle \, ds \, dt = 0, \forall \alpha \in \ker L^u, \forall \beta \in \ker L^v \right\}$$

Remarks:

1. Since we're taking a regular pair $(H, J)$, $L^u$ and $L^v$ are surjective because $u, v$ are solutions. They each have Fredholm index 1 so they have dim 1 kernels. Thus, since we know that $L^u(\frac{\partial u}{\partial s}) = 0$ and similarly for $v$, then

$$\ker L^u = \mathbb{R} \cdot \frac{\partial u}{\partial s}, \ker L^v = \mathbb{R} \cdot \frac{\partial v}{\partial s}$$

2. By exponential decay,

$$\sup \left( \left| \frac{\partial u}{\partial s} \right|, \left| \frac{\partial v}{\partial s} \right| \right) \leq K e^{-\delta |s|},$$

so we know vectors $\partial u/\partial s, \partial v/\partial s$ are in $L^q$ for every $q \geq 1$. Then, for all $Y \in W^{1,p}, \alpha \in \ker L^u, \beta \in \ker L^v$, $\langle Y, \alpha \#_\rho \beta \rangle \in L^1$ (by Cauchy-Schwarz).

3. The space $W^{1,p} = W_\rho \oplus W_\rho^\perp$. It follows from a general fact:

**Lemma 2.3.** *Let $1/p + 1/q = 1$. Let $E$ be a finite dimensional subspace of $W^{1,p} \cap L^q$. Then $W^{1,p} = E \oplus E^\perp$.*

*Proof.* It's clear that the two subspaces meet only at 0. By Hölder's inequality, since $E \subset L^p \cap L^q$, if $f \in E$, then $\|f\|_2^2 = \|f^2\|_1 \leq \|f\|_p \|f\|_q < \infty$. So $E \subset L^2$ which is a Hilbert space. So we can choose an ONB $\{e_1, ..., e_r\}$ of $E$. Every element $Z \in W^{1,p}$ satisfies $Z - \sum_i^r \langle e_i, Z \rangle e_i \in E^\perp$. □

The next proposition is important. It allows us to define $G_\rho$ and verify the conditions of the lemma.

**Proposition 2.4.** *There exists $C > 0$ such that for $\rho \geq \rho_0$, we have*

$$\forall Y \in W_\rho^\perp, \|L_\rho(Y)\|_p \geq C \|Y\|_{1,p}.$$

We begin by giving a list of consequences of this proposition. For its proof, see pp. 325-9.

1. Clearly, $\ker L_\rho \cap W_\rho^\perp = 0$.

2. Above, we said that $\text{Ind}(L_\rho) = 2$. So $\dim \ker L_\rho \geq 2$. Also, $\text{codim}\, W_\rho^\perp = \dim W_\rho \leq \dim(\ker L^u \times \ker L^v) = 2$. **Important point:** This dimension equals 2 because we have **transversality**. The first property makes it so that $\dim \ker L_\rho = 2$. This means that $L_\rho$ is surjective. Then $W^{1,p} = \ker L_\rho \oplus W_\rho^\perp$. Intuitively, we say that $\ker(d\mathcal{F})_{u\#_\rho v}$ is close to $\ker(d\mathcal{F})_u \#_\rho (d\mathcal{F})_v$ when $\rho$ is sufficiently large.

3. Consequently,

$$L_\rho : W_\rho^\perp \cong W^{1,p} / \ker L_\rho \to L^p = \text{Im } L_\rho$$

is bijective. Let $G_\rho$ be the right inverse.

4. $G_\rho$ is continuous because the proposition asserts:

$$\|G_\rho(Y)\|_{1,p} \leq C^{-1} \|L_\rho G_\rho(Y)\|_p = C^{-1} \|Y\|_p.$$

5. This proposition also gives us conditions (2) and (3) for the Newton-Picard Method. See pp. 324-5 for details proving this claim.

We then get a series of lemmas which help us to prove the continuity of $\gamma$ and also its differentiability with respect to $\rho$. A heuristic argument for this is, consider the parametrized Newton-Picard method. Then as long as the family varies smoothly and our approximations vary smoothly, then the solutions vary smoothly as well.

We can also prove that

$$\lim_{\rho\to+\infty} \|\gamma(\rho)\|_{1,p} = \lim_{\rho\to+\infty} \left\|\frac{\partial\gamma}{\partial\rho}\right\|_{1,p} = 0$$

This fact is useful in the proof to show that

$$\lim_{\rho\to+\infty} \hat{\psi}(\rho) = (\hat{u}, \hat{v}) \in \overline{\mathcal{L}}(x, z).$$

## 2.3 $\hat{\psi}$ is an Immersion

The last fact we stated above (Lemma 9.4.17) gives us that $\hat{\psi}$ is a proper map. For if $K \subset \mathcal{L}(x, z)$ is a compact set, then it is closed in particular and $\hat{\psi}^{-1}(K)$ must be closed (also, assume this to be connected for the moment). However, the limit of $\hat{\psi}(\rho)$ lies in the boundary and thus, there is a $\rho_1$ such that $[\rho_0, \rho_1] = \hat{\psi}^{-1}(K)$ which is compact. If the preimage is not connected, it must be a finite union of closed intervals which is still compact.

Furthermore, the image is closed. It is a result that an embedding with a closed image is precisely a proper injective immersion. Thus, we only need to show $\hat{\psi}$ is an injective immersion. To show that it's an immersion, we need to show that $\partial\psi/\partial\rho$ is not in the kernel of $d\pi$. The kernel of $(d\pi)_\psi$ is generated by $\partial\psi/\partial s$ as it is along $\mathbb{R}$ that we take the quotient. The picture is that we want $\psi$ to be transverse to the fibers of $\pi$.

Supposing that $\psi$ is not an immersion is equivalent to supposing there are sequences $(\rho_n)$ and $(\alpha_n)$ such that

$$\frac{\partial\psi}{\partial\rho}(\rho_n) = \alpha_n \frac{\partial\psi}{\partial s}(\rho_n).$$

Because $\psi(\rho)$ is close to $w_\rho$, we can deduce the following lemma:

**Lemma 2.5.** *The sequence $(\alpha_n)$ is bounded and we have*

$$\lim_{n\to+\infty} \left\| \left(\frac{\partial w_\rho}{\partial\rho} - \alpha_n \frac{\partial w_\rho}{\partial s}\right)_{\rho_n} \right\|_p = 0$$

Then, for $s \leq -1$, $w_\rho(s, t) = u(s + \rho, t)$, hence

$$\frac{\partial w_\rho}{\partial\rho} = \frac{\partial u}{\partial s}(s + \rho).$$

The lemma implies then that

$$\lim_{n\to+\infty} \left\|\frac{\partial u}{\partial s}(s + \rho_n, t) - \alpha_n \frac{\partial u}{\partial s}(s + \rho_n, t)\right\|_{L^p((-\infty,-1]\times S^1)} = 0.$$

This means that $\alpha_n \to 1$. But if we consider the values for $s \geq 1$, we get

$$\lim_{n\to+\infty} \left\|-\frac{\partial v}{\partial s}(s - \rho_n, t) - \alpha_n \frac{\partial v}{\partial s}(s - \rho_n, t)\right\|_{L^p([1,\infty)\times S^1)} = 0$$

which implies that $\alpha_n \to -1$. This is the contradiction we need. Therefore, $\hat{\psi}$ is an immersion for large $\rho$.

It is also injective: its image is contained in a connected component of the manifold $\mathcal{L}(x,z)$ (dim = 1) which is not compact. This component is diffeomorphic to an open interval $I \cong \mathbb{R}$. By Rolle's Theorem, if $\hat{\psi} : \mathbb{R} \to \mathbb{R}$ is not injective, then $\partial\hat{\psi}/\partial\rho = 0$ for some $\rho$. But $\hat{\psi}$ is an immersion so this never happens.

## 2.4 Uniqueness of the Gluing

In the previous construction of $\hat{\psi}$, we used an approximate solution (aka pre-gluing) $w_\rho$ which equaled, for $s \geq 1$, $v(s-\rho, t)$. However, we could replace this expression with $v(s - \nu(\rho), t)$ where $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function increasing to $+\infty$. We can show that if construct a pre-gluing $w_{\nu,\rho}$ and an associated $\hat{\psi}_\nu : [\rho_\nu, \infty) \to \mathcal{L}(x,z)$, it will also be an injective immersion and $\lim \hat{\psi}_\nu = (\hat{u}, \hat{v})$. That is, it has the same relevant properties as when $\nu = \mathrm{id}$. Naturally, we ask: "Does the gluing depend on $\nu$?"

We would ultimately like to show that if $\ell_n$ is a sequence converging to the broken trajectory $(\hat{u}, \hat{v})$, then for large $n$, it will be contained in Im $\hat{\psi}$. It turns out that we can realize the $\ell_n$ as arising from a pre-gluing $w_{\nu,\rho}$ and we're able to then show that for large $n$, the $\ell_n$ are contained in Im $\hat{\psi}_\nu$.

If we also know that for some large enough $\rho_\nu$, Im $\hat{\psi}_\nu \subset$ Im $\hat{\psi}$, we're able to conclude that the $\ell_n$ are eventually contained in Im $\hat{\psi}$ and thus, there is uniqueness in gluing. The authors take these in steps, starting with what we just stated.

1. Prove that if $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ has the properties mentioned above, then there is some $\rho_\nu$ such that for $\rho \geq \rho_\nu$, Im $\hat{\psi}_\nu \subset$ Im $\hat{\psi}$.

2. Suppose that $\ell_n \to (\hat{u}, \hat{v})$ in $\overline{\mathcal{L}}(x,z)$. We wish to prove the following proposition:

   **Proposition 2.6** (9.4.3)**.** *There exists a lift $\tilde{\ell}_n \in \mathcal{M}(x,z)$ of $\ell_n$ and a smooth increasing function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $(s,t) \in \mathbb{R} \times S^1$, $\tilde{\ell}_n(s,t) = \exp_{w_{\nu,\rho_n}(s,t)} Y_n(s,t)$, where $\rho_n \to +\infty$ and $Y_n \in w^*_{\nu,\rho_n} TM$ satisfies* $\lim_{n\to\infty} \|Y_n\|_\infty = 0$.

   Observe that this result is saying that for **any** $\ell_n \to (\hat{u}, \hat{v})$, there are lifts constructed using $\nu$, a function with nice properties. I found the proofs in this step to be rather technical, even if the theory wasn't hard: mostly playing around with sequences as in a real analysis class.

3. Next, we prove that the $Y_n$ from above satisfy $Y_n \in W^{1,p}(w^*_{\nu,\rho_n} TM)$ for all $p > 2$ and also $\lim \|Y_n\|_{1,p} = 0$. Since these $\tilde{\ell}_n = \exp Y_n$ are solutions of the Floer equation with finite energy, they satisfy an exponential decay property. Then, we can use some results from ch. 8 concerning exponential decay of $C^2$ solutions to the linearized Floer equation along a solution. The exponential decay results will allow us to conclude Lemma 9.6.13: for all $(s,t) \in \mathbb{R} \times S^1$

   $$\max\left\{\|Y_n(s,t)\|, \left\|\frac{\partial Y_n}{\partial s}(s,t)\right\|, \left\|\frac{\partial Y_n}{\partial t}(s,t)\right\|\right\} \leq K e^{-\delta|s|}.$$

   From this, we can conclude that $Y_n \in W^{1,p}$ because

   $$K^p \int_{-\infty}^{\infty} e^{-\delta p|s|}\, ds = 2K^p \int_0^\infty e^{-\delta p s}\, ds = \frac{2K^p}{\delta p} < \infty.$$

   To obtain the result that $\|Y_n\|_{1,p} \to 0$, the authors use some technical lemmas whose proofs are saved for ch. 13.

4. So we have a sequence $\tilde{\ell}_n$ of the same form as solutions produced by the Newton-Picard method. We would like to see that for $n$ large enough, that the $Y_n$ in fact, are identical with solutions produced by Newton-Picard. To do this, we will need to generalize the method slightly to work when we vary $n$.

   Let $F = \mathcal{F} \circ \exp$ and $L = dF$. Recall that before, we gave a decomposition $W^{1,p} = \ker L \oplus W_{\rho_n}^{\perp}$ and looked for a solution in a slice. We'll do the same here by giving a decomposition and then use a contracting map $\varphi_n : h_n + W_{\rho_n}^{\perp} \to h_n + W_{\rho_n}^{\perp}$ where $h_n \in B(0, \epsilon_0) \cap \ker L$. The contracting map will have a unique fixed point $\gamma_n(h)$ which is a solution in the sense that $F\gamma_n(h) = 0$; so $\exp_{w_\nu, \rho_n} \gamma_h(h) \in \mathcal{M}(x, z)$.

   As promised, for large $n$, $Y_n = \gamma_h(h)$; the proof of this claim relies on knowing that $\lim \|Y_n\|_{1,p} = 0$. With this knowledge in mind, we're able to say more about our solutions $Y_n$. What is most germane is that the map $h \mapsto \gamma_n(h)$ is continuous. Having continuity of this map, we can form a connectedness argument to finally conclude that for large $n$, $\ell_n = \pi \circ \tilde{\ell}_n \in \mathrm{Im}\ \hat{\psi}_\nu$. By step 1, then for larger $n$, $\ell_n \in \mathrm{Im}\ \hat{\psi}$.

Remark: The proof for continuity of $h \mapsto \gamma_n(h)$ is fun. Let $\overline{F} : \ker L \oplus W^{\perp} \to L^p$, defined by $\overline{F}(h, w) = F(h + w)$. Then $\overline{F}(h, \gamma_n(h) - h) = 0$ because $F\gamma_n(h) = 0$. Showing that $(d\overline{F})_{\gamma_n(h)}$ is invertible will allow us to apply the Implicit Function Theorem to conclude that $h \mapsto \gamma_n(h) - h$ is a continuous map and therefore, so is $h \mapsto \gamma_n(h)$. As a reminder, we'll state the Implicit Function Theorem from multivariate calculus but applies for manifolds and Banach spaces.

**Theorem 2.7** (Implicit Function Theorem)**.** *Let $f : \mathbb{R}^n \oplus \mathbb{R}^m \to \mathbb{R}^m$ be smooth. Suppose that $f(x_0, y_0) = 0$ where $x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m$. Then, if $(df)_{(x_0,y_0)}$ is invertible, there exists a neighborhood of $(x_0, y_0)$, call it $U \subset \mathbb{R}^n$, and a **continuous** map $g : U \to \mathbb{R}^m$, such that $g(x_0) = y_0$ and for $x \in U$, $f(x, g(x)) = 0$.*