Ch. 8 Outline: Linearization and Transversality

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This is an outline of ch. 8 of *Morse Theory and Floer Homology.* "Proofs" are more of sketches.

Let (M, ω) be our closed symplectic manifold. We will, at many points, need to consider TM but we prefer to look locally at $(U \subset M, \omega) \cong (\mathbb{R}^{2n}, \omega_0)$. In this case, we can then take a local trivialization of the tangent bundle and consider $U \times \mathbb{R}^{2n}$ as our tangent bundle.

We begin with some brief remarks about the spaces we'll be dealing with. We need to consider the Sobolev space $W^{1,p}(\mathbb{R} \times S^1, M)$: it may be viewed as the completion of $C^\infty(\mathbb{R} \times S^1, M)$ under the norm $\| \cdot \|_{1,p}$. Take $Y \in W^{1,p}(\mathbb{R} \times S^1, TM)$. Then

$$\|Y\|_{1,p} := \left(\int_{\mathbb{R} \times S^1} |Y|^p + \left| \frac{\partial Y}{\partial s} \right|^p + \left| \frac{\partial Y}{\partial t} \right|^p ds\, dt \right)^{1/p} < \infty.$$

The best point of view to take here is that elements of $W^{1,p}$ are distributions so that they have derivatives in the sense of distributions. But in general, if we have $W^{k,p}(\mathbb{R}^n)$, we need $kp > n = \dim(\text{domain})$ in order for the elements to be continuous (Sobolev/Reillich Theorems). Here, the space is $\mathbb{R} \times S^1$ so $n = 2$. Thus, we would like $p > 2$. Another complication is that the variable s is varying over the noncompact space $\mathbb{R}$.

The second space of interest is that of $C^\infty_\epsilon(H_0)$. We fix a nondegenerate Hamiltonian H_0; the space $C^\infty_\epsilon(H_0)$ is all the perturbations of H_0 by some $h : M \to \mathbb{R}$ such that $H = H_0 + h$ has the same periodic orbits and is also nondegenerate. The point is that we wish to perturb the Floer equation so that $\mathcal{M}(x, y, J, H)$ is a manifold of dimension $\mu(x) - \mu(y)$. In general, a given Hamiltonian does not produce such a manifold but arbitrarily small perturbations will show the moduli space of solutions to be a manifold.

1 The Main Theorems

1.1 Spaces We'll Work With

Recall that the space $\mathcal{P}(x, y)$ is comprised of maps of the form $(s, t) \mapsto \exp_{w(s,t)} Y(s, t)$ where $Y \in W^{1,p}(w^*TM)$ and $w \in C^\infty_\searrow(x, y)$. The definition of $C^\infty_\searrow(x, y)$ is as follows: it consists of maps $u : \mathbb{R} \times S^1 \to M$ such that u limits to periodic orbits x and y and there exist $K, \delta > 0$ such that

$$\left| \frac{\partial u}{\partial s}(s, t) \right| \leq K e^{-\delta |s|} \qquad \left| \frac{\partial u}{\partial t}(s, t) - H_t(u) \right| \leq K e^{-\delta |s|}.$$

Consider the **fiber** bundle $\mathcal{E} \to \mathcal{P}(x, y) \times C^\infty_\epsilon(H_0)$ with the total space defined as $\mathcal{E} = \{(u, h, Y) : Y \in L^p(u^*TM)\}$. Let $\mathcal{E}_0$ be the zero section. Let σ be a section of this bundle which sends

$$(u, h) \mapsto \frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} + \nabla_u (H_0 + h).$$

The differential is $(d\sigma)_{(u,h)}(Y,\eta) = (d\mathcal{F})_u(Y) + \nabla_u\eta$; the extra $\nabla_u\eta$ comes from the h we added to obtain $H = H_0 + h$. Note also that $\sigma^{-1}(\mathcal{E}_0) = \mathcal{Z}(x,y,J) = \{(u,H) : h \in C^\infty_\epsilon(H_0), u \in \mathcal{M}(x,y,J,H)\}$ as sets. If σ is transverse to $\mathcal{E}_0$, then we would have that $\sigma^{-1}(\mathcal{E}_0) = \mathcal{Z}(x,y,J)$ is a manifold by the Implicit Function Theorem. This transversality condition is equivalent to a certain projection of $(d\sigma)_{(u,h)}$ being surjective. The projection is from $T_{\sigma(u,h)}\mathcal{E}$ onto the tangent space of the **fiber** at $\sigma(u,h) \in \mathcal{E}_0$. Call it Π.

1.2 Theorems

Theorem 1.1 (8.1.5)**.** *For every nondegenerate Hamiltonian H, every almost complex structure J compatible with ω, and every $u \in \mathcal{M}(x,y,J,H)$, $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.*

The first theorem says that $\mathcal{F}$ restricted to $\mathcal{M}(x,y,J,H)$ is a Fredholm map as the differential at each point is a Fredholm operator. Note that the index is independent of u in the moduli space.

Fix ACS J. Let $\mathcal{Z}(x,y,J)$ be the space of solutions connecting x and y for all Floer maps corresponding to the different perturbations of H_0. The next theorem states:

Theorem 1.2 (8.1.4)**.** *Let $(u,H) \in \mathcal{Z}(x,y,J)$, where $H = H_0 + h$ and $\mathcal{F}^H$ is the corresponding Floer operator. Then*

$$\begin{aligned} \Gamma : W^{1,p}(\mathbb{R}\times S^1, \mathbb{R}^{2n}) \times C^\infty_\epsilon(H_0) &\to L^p(\mathbb{R}\times S^1, \mathbb{R}^{2n}) \\ (Y,h) &\mapsto (d\mathcal{F}^H)_u(Y) + \nabla_u h \end{aligned}$$

is surjective and admits a continuous right inverse.

In this second theorem, Γ is the composition of $\Pi \circ d\sigma$. We prove this in section 3. A corollary is:

Theorem 1.3 (8.1.3)**.** *$\mathcal{Z}(x,y,J)$ is a Banach manifold.*

This theorem follows immediately from the Implicit Function Theorem and our prior results. Now that we know that $\mathcal{Z}(x,y,J) \subset \mathcal{P}(x,y) \times C^\infty_\epsilon(H_0)$ is a manifold, we wish to understand its submanifolds $\mathcal{M}(x,y,J,H)$. Let $\pi : \mathcal{Z}(x,y,J) \to C^\infty_\epsilon(H_0)$ be the projection map $(u,h) \mapsto h$. It is smooth. Setting $H = H_0 + h$, the tangent map is

$$(d\pi)_{(u,H)} : T_{(u,H)}\mathcal{Z}(x,y,J) \to T_h C^\infty_\epsilon(H_0) = C^\infty_\epsilon(H_0)$$

which maps $(Y,\eta) \mapsto \eta$; $(d\pi)_{(u,H)}$ is surjective. In fact, this tangent space $T_{(u,H)}\mathcal{Z}(x,y,J) = \ker(d\sigma)_{(u,h)}$ because $\sigma^{-1}(\mathcal{E})_0 = \mathcal{Z}(x,y,J)$. Then one can see that $\ker(d\pi)_{(u,H_0+h)}$ consists of elements $(Y,0)$ where $(d\mathcal{F})_u Y = 0$. Thus, $\ker(d\pi)_{(u,H)} = \ker(d\mathcal{F})_u$.

Let's write $L = (d\mathcal{F})_u$. Grant, for now, that $\mathcal{F}$ is a Fredholm map. Then $\ker L = \ker(d\pi)_{(u,H)}$ have finite dimension. Similarly, if $T : C^\infty_\epsilon(H_0) \to L^p(u^*TM)$ is the map sending $\eta \mapsto \nabla_u\eta$, then Im $(d\pi)_{(u,H)} = T^{-1}(\text{Im } L)$. This follows from the observation that since (Y,η) satisfy $LY + \nabla\eta$, being in $\ker d\sigma$, $LY = -\nabla_u\eta$ and so $T^{-1}(LY) = \eta = d\pi(Y,\eta)$. Since the cokernel of L has finite dimension, so does $(d\pi)_{(u,H)}$. Here's a diagram that hopefully illuminates some of this discussion.

$$\begin{array}{ccccc} & & & & \mathcal{E} \\ & & & & \downarrow\uparrow \sigma \\ \mathcal{M}(x,y,J,H) & \hookrightarrow & \mathcal{Z}(x,yJ) & \hookrightarrow & \mathcal{P}(x,y \times C^\infty_\epsilon(H_0) \\ & & \downarrow \pi & & \\ & & C^\infty_\epsilon(H_0) & \xrightarrow{T} & L^p(u^*TM) \end{array}$$

By the Sard-Smale theorem, since π is Fredholm, the set of regular values of π is dense in $C^\infty_\epsilon(H_0)$. Then take an $H \in \mathcal{H}_{\text{reg}}$. We will find that $\pi^{-1}(H) = \mathcal{M}(x, y, J, H)$ is a submanifold and has dimension $\mu(x) - \mu(y)$. The next two theorems summarize what we just said.

Theorem 1.4 (8.1.1)**.** *Let H_0 be a fixed nondegenerate Hamiltonian. There exists a neighborhood of $0 \in C^\infty_\epsilon(H_0)$ and a countable intersection of dense open subsets $\mathcal{H}_{reg}$ in this neighborhood such that if $h \in \mathcal{H}_{reg}$, then $H = H_0 + h$ is nondegenerate and the map $(d\mathcal{F})_u$ is surjective for every $u \in \mathcal{M}(x, y, J, H)$.*

Theorem 1.5 (8.1.2)**.** *For every $h \in \mathcal{H}_{reg}$ and for all contractible orbits x and y of period 1 of H_0, $\mathcal{M}(x, y, J, H_0 + h)$ is a manifold of dimension $\mu(x) - \mu(y)$.*

2 Linearization

Linearization of any differential equation usually amounts to finding a map L and considering solutions to $Lu = 0$. In our case, we have the Floer map $\mathcal{F}$; because we want to consider Banach spaces, we extend to the larger space $W^{1,p}$.

We'll take $L = (d\mathcal{F})_u(Y) = (\bar{\partial} + S)Y$ where $\lim_{s\to\pm\infty} S(s, t) = S^\pm(t)$ uniformly in t; $S^\pm$ are symmetric operators.

3 Transversality

As mentioned, the transversality condition is equivalent to showing that Γ is surjective. Thus, let us suppose that Γ is not surjective. By linear algebra, the image of Γ is a closed subspace. Then by the Hahn-Banach theorem, there exists a functional $\varphi : L^p \to \mathbb{R}$ such that $\varphi|_{\operatorname{Im}\Gamma} = 0$. By the Riesz Representation theorem, if q satisfies $1/p + 1/q = 1$, there exists a nonzero $Z \in L^q(\mathbb{R} \times S^1; \mathbb{R}^{2n})$ such that $\varphi(Y) = \langle Y, Z\rangle$ where this pairing is understood to be

$$\langle Y, Z\rangle = \int_{\mathbb{R}\times S^1} \langle Y(s, t), Z(s, t)\rangle \, ds \, dt.$$

Lemma 3.1 (8.5.1)**.** *Let Z be as above. Z is in fact of class C^∞ and for every $h \in C^\infty_\epsilon(H_0)$ and for every $Y \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$, we have*

$$\langle Z, (d\mathcal{F})_u(Y)\rangle = 0 \text{ and } \langle Z, \nabla_u h\rangle = 0.$$

Proof. Since $Z \perp \operatorname{Im}\Gamma$, then when $Y = 0$, this implies $\langle Z, (d\mathcal{F})_u Y + \nabla_u h\rangle = \langle Z, \Pi \circ \nabla_u h\rangle = 0$. Similarly, we see that $\langle Z, (d\mathcal{F})_u Y\rangle = 0$ when we let h be constant.

To see that Z is smooth, we consider the adjoint of $L := \bar{\partial} + S$, which is just $d\mathcal{F}$: $L^* = -\partial/\partial s + J_0 \partial/\partial t + {}^t S$. Then, $\forall Y \in W^{1,p}$, $0 = \langle LY, Z\rangle = \langle Y, L^* Z\rangle$. This means that $L^* Z = 0$. Since L^* is elliptic, Z must be of class C^∞. □

The next lemma tells us more about this vector field Z.

Lemma 3.2 (8.5.3)**.** *If $Z \in L^q$ is of class C^∞ and $\langle Z, \nabla_u h\rangle = 0$ for all $h \in C^\infty_\epsilon(H_0)$, then there is a C^∞ function $\lambda : S^1 \to \mathbb{R}$ such that $Z(s, t) = \lambda(t)\frac{\partial u}{\partial s}$.*

Proof. The start of the proof requires the notion that a solution u be "somewhere injective." The result we ultimately want from this "somewhere injectivity" is for the regular points of u—call the set $R(u)$—to form a dense open set in $\mathbb{R} \times S^1$. We'll show this somewhere injective result later. These regular points are not quite defined the usual way. A point (s_0, t_0) is **regular** if it is not a critical point **and** also satisfies $u(s_0, t_0) \neq u(s, t_0)$ for all $s \in \mathbb{R} \cup \{\pm\infty\}$ (note that t_0 appears in both).

Granted this density result, we show that Z and $\partial u/\partial s$ are linearly dependent. If they were independent, we may construct an $h \in C^\infty_\epsilon(H_0)$ such that $\langle Z, \nabla_u h\rangle \neq 0$; this is a contradiction. So then the two are linearly dependent, meaning $Z(s,t) = \lambda(s,t)\frac{\partial u}{\partial s}$ on $R(u)$ for some $\lambda : R(u) \to \mathbb{R}$. But $R(u)$ is dense so we may extend λ to $\mathbb{R} \times S^1$.

Lastly, we show that λ is independent of s. If $\partial\lambda/\partial s \neq 0$ for some (s_0, t_0), we can again construct a perturbation h such that $\langle Z, \nabla_u h\rangle \neq 0$. Thus, $\lambda(t)$ is independent of s. □

We continue with the proof to show Γ is surjective. If Γ is not surjective, we can produce this $Z(s,t) = \lambda(t)\frac{\partial u}{\partial s}$.

1. First, we show that $\lambda(t) \neq 0$ for any t. If not, we have a t_0 such that $\lambda(t_0) = 0$ so then $Z(s, t_0) = 0$ for all s. Then for all $k \in \mathbb{Z}_{\geq 0}$, $\partial^k Z/\partial s^k = 0$. But $0 = L^* Z = -\partial Z/\partial s + J_0 \partial Z/\partial t + {}^t S(s,t) Z$; at (s, t_0), the first and last term equal zero means $J_0 \partial Z/\partial t = 0$ implying that $\partial Z/\partial t = 0$. By induction, then **all** derivatives of Z as well as Z itself vanish on $\mathbb{R} \times \{t_0\}$. Z is a solution of a perturbed Cauchy-Riemann equation and it and **all its derivatives vanish** on this line. By the Continuation Principle, in fact, $Z = 0$ everywhere. This is the contradiction we need.

2. We may then assume $\lambda(t) > 0$. Then, we may define a function in terms of s:

$$f(s) = \int_0^1 \left\langle \frac{\partial u}{\partial s}(s,t), Z(s,t) \right\rangle dt = \int_0^1 \lambda(t) \left| \frac{\partial u}{\partial s}(s,t) \right|^2 dt > 0, \forall s \in \mathbb{R}.$$

   Since $\partial u/\partial s \to 0$ as $s \to \pm\infty$, $f(s)$ tends to zero. If we show that f is constant, we would get a contradiction. Thus, we aim to show f is constant.

   Recall that $Y = \partial u/\partial s$ is a solution of $LY = 0$ by the following argument: if u is a solution of the Floer equation, then so is its translates $u \cdot s$. Hence, $\mathcal{F}(u \cdot s) = 0$ and

$$0 = \frac{d}{ds}\mathcal{F}(u \cdot s) = (d\mathcal{F})_u \left(\frac{\partial u}{\partial s}\right).$$

   Also, $L^* Z = 0$. Thus, we get the following relations

$$\frac{\partial Y}{\partial s} = -J_0 \frac{\partial Y}{\partial t} - SY \text{ and } \frac{\partial Z}{\partial s} = J_0 \frac{\partial Z}{\partial t} + {}^t SZ.$$

   The derivative of f is

$$\begin{aligned}
\frac{d}{ds}\int_0^1 \langle Y, Z\rangle\, dt &= \int_0^1 \left(\left\langle \frac{\partial Y}{\partial s}, Z \right\rangle + \left\langle Y, \frac{\partial Z}{\partial s} \right\rangle \right) dt \\
&= \int_0^1 \left(\left\langle -J_0 \frac{\partial Y}{\partial t}, Z \right\rangle - \langle SY, Z\rangle + \left\langle Y, J_0 \frac{\partial Z}{\partial t} \right\rangle + \langle Y, {}^t SZ\rangle \right) dt \\
&= -\int_0^1 \left(\left\langle J_0 \frac{\partial Y}{\partial t}, Z \right\rangle + \left\langle J_0 Y, \frac{\partial Z}{\partial t} \right\rangle \right) dt \\
&= -\int_0^1 \frac{\partial}{\partial t}\langle J_0 Y, Z\rangle\, dt = 0.
\end{aligned}$$

   The last line holds because $\langle J_0 Y, Z\rangle = \lambda(t)\langle J\frac{\partial u}{\partial s}, \frac{\partial u}{\partial s}\rangle = \lambda(t)\omega(J\frac{\partial u}{\partial s}, J\frac{\partial u}{\partial s}) = 0$. Thus, f is constant and not going to 0. This is our final contradiction which allows us to conclude that Γ must be surjective.

Γ has a continuous right inverse from the following abstract lemma:

Lemma 3.3 (8.5.6)**.** *Let $E, F,$ and G be Banach spaces and*

$$L_1 : E \to G, \quad L_2 : F \to G$$

be linear operators. Assume that L_1 is Fredholm and that $\Gamma : E \oplus F \to G$ defined by $\Gamma(x, y) = L_1(x) + L_2(y)$, is surjective. Then Γ admits a continuous right inverse.

Proof. Write $G = \text{Im } (L_1) \oplus H$, where H is closed and finite-dimensional. Let E' be a closed subspace of E such that $E = \ker L_1 \oplus E'$. Clearly $L_1 : E' \to \text{Im } L_1$ is bijective. Let L_1^{-1} denote the composition $L_1^{-1} : \text{Im } L_1 \to E' \subset E \subset E \oplus F$.

Let $h_1, ..., h_r$ be a basis of H and $x_1, ..., x_r \in E \oplus F$ be such that $\Gamma(x_i) = h_i$. Define $\nu : H \to E \oplus F$ by $\nu(h_i) = x_i$. This is a continuous map, since its image is finite-dimensional. Now the map

$$\Pi : \text{Im } L_1 \oplus H \to E \oplus F; (z, h) \mapsto (L_1^{-1}(z), 0) + \nu(h)$$

is a right inverse of Γ as $\Gamma \circ \Pi = \text{id}$ is easy to check. Moreover, Π is continuous since it can be written as $\Pi = (L_1^{-1} \circ pr_{\text{Im } L_1}, 0) + \nu \circ pr_H$. □

We remarked earlier that this immediately gives us that $\mathcal{Z}(x, y, J)$ is a Banach manifold. Also, we had shown that $\pi : \mathcal{Z}(x, y, J) \to C_\epsilon^\infty(H_0)$ is a Fredholm map. Then, we can apply the Sard-Smale theorem.

Theorem 3.4 (Sard-Smale; 8.5.7)**.** *Let E and F be two separable Banach spaces, let $U \subset E$ be open and let $L : U \to F$ be a smooth Fredholm map. Then the set of regular values of L is a countable intersection of dense open subsets.*

Remark: The separability is essential. It guarantees that we can extract a countable subcover from any open cover. This way, we can obtain a countable intersection of dense open subsets. Baire's Category theorem then says that the intersection is also dense because E is a complete metric space.

To obtain Theorem 1.4, we prove a lemma which immediately implies the theorem.

Lemma 3.5 (8.5.9)**.** *The regular values of π are exactly the $h \in C_\epsilon^\infty(H_0)$ such that for every $u \in \mathcal{M}(x, y, J, H_0 + h)$, the map $(d\mathcal{F})_u$ is surjective.*

Proof. Let h be a regular value of π and u a solution for the Floer equation with $H = H_0 + h$. If $(d\mathcal{F})_u$ is not surjective, then there exists a vector field $Z \in L^q$ such that $\forall Y \in W^{1,p}$, $\langle (d\mathcal{F})_u(Y), Z \rangle = 0$.

Now, $(d\pi)_{(u,H)}$ is surjective ($H = H_0 + h$). By the discussion above, following after Theorem 1.3, we see that for every $\eta \in C_\epsilon^\infty(H_0)$, there exists a vector field Y such that $LY + \nabla_u \eta = 0$. This implies that $\langle Z, \nabla_u \eta \rangle = 0$. The proof of the surjectivity of Γ, in particular the part dealing with the existence of Z, also showed that $Z = 0$, Thus, the only thing orthogonal to the image of $(d\mathcal{F})_u$ iz zero, which means it must be surjective.

Conversely, given h, if $(d\mathcal{F})_u$ is surjective for every $u \in \mathcal{M}(J, H_0 + h)$, let's show any given $\eta \in C_\epsilon^\infty(H_0)$ is in the image of $d\pi$. Choose $Y \in W^{1,p}$ such that $(d\mathcal{F})_u(Y) = -\nabla_u \eta$. Then, (Y, η) is in $\ker d\sigma = T_{(u,H)}\mathcal{Z}(x, y, J)$, the domain of $d\pi$. Our chosen pair (Y, η) satisfies $d\pi(Y, \eta) = \eta$ and so $(d\pi)_{(u,H)}$ is surjective, implying that h is a regular value. □

With Sard-Smale, we have proven Theorem 1.4 (admitting "somewhere injective"). Let us prove the last main Theorem 1.5.

Proof. Let h be a regular value of $\mathcal{F}$. Lemma 3.5 says that it is also a regular value of π. Consequently, $\pi^{-1}(h)$ is a manifold and its dimension equals the Fredholm index of π which is

$$\begin{aligned}\dim\ker(d\pi)_{(u,H)} &= \dim\ker(d\mathcal{F})_u \\ &= \operatorname{Ind}(d\mathcal{F})_u \\ &= \mu(x) - \mu(y).\end{aligned}$$

The elements of $\pi^{-1}(h)$ are solutions in $\mathcal{P}^{1,p}$ (see p. 227 for a reminder of the definition of this space). Elliptic regularity gives that $\pi^{-1}(h) \subset \mathcal{M}(x, y, J, H_0 + h)$. A proposition in the book gives that $\mathcal{M}(x, y) \subset \mathcal{P}^{1,p}$. Thus, $\mathcal{M}(x, y, J, H) \subset \pi^{-1}(h)$ and so $\pi^{-1}(h) = \mathcal{M}(x, y, J, H_0 + h)$. The tangent space $T_{(u,h)}\mathcal{M}(x, y, J, H_0 + h) = \ker d\pi_{(u,h)}$. □

4 The Solutions of the Floer Equation are "Somewhere Injective"

We now discuss this notion of "somewhere injective" as well as the continuation principle. We begin with a proposition which is something of a trick to turn our perturbed equation into a Cauchy-Riemann equation.

Proposition 4.1 (8.6.1)**.** *Let $u : \mathbb{R} \times S^1 \to M$ be a solution of the equation*

$$\frac{\partial u}{\partial s} + J(t, u)\left(\frac{\partial u}{\partial t} - X(t, u)\right) = 0$$

So X is periodic in t. There exists an almost complex structure $\widetilde{J}$ and a diffeomorphism φ on M, as well as a smooth map $v : \mathbb{R}^2 \to M$ such that

$$\frac{\partial v}{\partial s} + \tilde{J}\frac{\partial v}{\partial t} = 0$$

$$v(s, t + 1) = \varphi(v(s, t))$$

and for $(s, t) \in \mathbb{R} \times [0, 1)$. $C(u) = C(v)$ and $R(u) = R(v)$. So u and v have the same critical points and also the same regular points which do not admit multiples; i.e. there is no pair of points $(s, t), (s', t)$ (same t!) such that $v(s, t) = v(s', t)$.

Proof. $M \times S^1$ is compact so we have a family of isotopies (almost a flow) ψ_t of X_t defined for all of M. Let $v(s, t) = \psi_t^{-1}(u(s, t))$. Then

$$\frac{\partial u}{\partial s} = (d\psi_t)\left(\frac{\partial v}{\partial s}\right); \frac{\partial u}{\partial t} = (d\psi_t)\left(\frac{\partial v}{\partial t}\right) + X_t(u).$$

Consequently,

$$\begin{aligned}0 &= \frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_t(u)\right) \\ &= (d\psi_t)\left(\frac{\partial v}{\partial s}\right) + J(u)(d\psi_t)\left(\frac{\partial v}{\partial t}\right) \\ &= (d\psi_t)\left(\frac{\partial v}{\partial s} + (d\psi_t)^{-1}J(u)(d\psi_t)\left(\frac{\partial v}{\partial t}\right)\right)\end{aligned}$$

Let $\psi_t^* J(v) := (d\psi_t)^{-1}J(u)(d\psi_t)$. Then

$$\frac{\partial v}{\partial s} + \psi_t^* J(v) \frac{\partial v}{\partial t} = 0$$

Let $\varphi = \psi_1$ and $\widetilde{J} = \psi_1^* J(v)$. The other properties are easy to verify. φ is a diffeomorphism so the critical and regular values u remain unchanged. And $\varphi = \psi_1$ is 1-periodic. □

The main proposition of this section is:

Proposition 4.2 (8.6.3)**.** *Let v be a smooth solution of the Cauchy-Riemann equation with respect to J (we'll rename $\widetilde{J}$ from before and just call it J) satisfying the periodicity condition $v(s, t + 1) = \varphi(v(s, t))$, and such that $\partial v/\partial s \not\equiv 0$. Then $R(v)$ is an open dense subset of $\mathbb{R}^2$.*

Remark: That $R(v)$ is open is easy since being a critical value is a closed condition as is injectivity; so $R(v)$ is defined by open conditions. However, density is difficult. At one point in the proof, we need the Continuation Principle for the perturbed Cauchy-Riemann equation:

$$\frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S \cdot Y = 0.$$

Proposition 4.3 (Continuation Principle; 8.6.6)**.** *Let Y be a solution of the perturbed Cauchy-Riemann equation on an open subset $U \subset \mathbb{R}^2$. Then the set C of points $(s, t) \in U$ such that Y has an infinite order at (s, t) is open and closed in U. If U is connected and Y is zero on a nonempty open subset of U, then Y is identically zero on U.*

The Continuation Principle is a consequence of the following lemma:

Lemma 4.4 (Similarity Principle; 8.6.8)**.** *Let $Y : B_\epsilon \to \mathbb{C}^n$ be a smooth solution of the perturbed Cauchy-Riemann equation; let $p > 2$. Then there is a positive $\delta < \epsilon$, $A \in W^{1,p}(B_\delta, GL(\mathbb{R}^{2n}))$, and holomorphic map $\sigma : B_\delta \to \mathbb{C}^n$ s.t. for all $(s, t) \in B_\delta$, $Y(s, t) = A(s, t)\sigma(s + it)$ and $J_0 A(s, t) = A(s, t) J_0$; i.e. A is $\mathbb{C}$-linear.*

Remark: We can actually assume $Y \in W^{1,p}$ and $S \in L^p(B_\epsilon, End_{\mathbb{R}}(\mathbb{R}^{2n}))$ for $p > 2$.

Let's use the Similarity Principle to prove the continuation principle.

Proof. The set C of infinite order zeros of Y is closed; if (s_k, t_k) is a sequence of infinite order zeros of Y converging to (s, t), since $p > 2$, Y is continuous and then (s, t) is an infinite order zero of Y.

Let $z_0 \in C$. By the Similarity Principle, $Y(z) = A(z)\sigma(z)$ on some $B_\delta(z_0)$. Every point of $B_\delta(z_0)$ is an infinite order zero of Y if and only if every point is an infinite order zero of σ. Now

$$\sup_{|z-z_0|\le r} |\sigma(z)| = \sup_{|z-z_0|\le r} |A^{-1}(z)Y(z)| \le K \sup_{|z-z_0|\le r} |Y(z)|.$$

The last inequality holds because A is continuous and invertible (again, $p > 2$); by the Open Mapping Theorem, A^{-1} is also continuous and has operator norm K. Then

$$\lim_{r\to 0} \frac{\sup_{|z-z_0|\le r} |\sigma(z)|}{r^k} \le K \lim_{r\to 0} \frac{\sup_{|z-z_0|\le r} |Y(z)|}{r^k} = 0.$$

The equality with 0 is just what it means for Y to have an infinite order zero at z_0. So then z_0 is an infinite order zero for σ. But σ is holomorphic and thus analytic; therefore, it must be that $\sigma \equiv 0$ on $B_\delta(z_0)$. Hence, $B_\delta(z_0) \subset C$. So C is also open. □

How do we prove the Similarity Principle? To establish that A is $\mathbb{C}$-linear and σ is holomorphic, we need the following theorem:

Theorem 4.5 (8.6.11)**.** *For $p > 1$, $\bar{\partial} : W^{1,p}(S^2, \mathbb{C}^n) \to L^p(\bigwedge^{0,1} T^*S^2 \otimes \mathbb{C}^n)$ is a surjective Fredholm operator.*

Proof. If we grant the surjectivity of $\bar{\partial}$, then it's easy to show it is Fredholm. The index would equal $\dim \ker \bar{\partial}$ if it is finite. Now if $\bar{\partial} Y = 0$, then

$$\frac{\partial Y}{\partial s} = -J_0 \frac{\partial Y}{\partial t}$$

which means $\|\frac{\partial Y}{\partial s}\| = \|\frac{\partial Y}{\partial t}\|$. Recall that $g(\frac{\partial Y}{\partial s}, \frac{\partial Y}{\partial s}) = \omega(\frac{\partial Y}{\partial s}, J_0 \frac{\partial Y}{\partial s}) = \omega(\frac{\partial Y}{\partial s}, \frac{\partial Y}{\partial t})$. Let $S^2 = \mathbb{C} \cup \{\infty\}$; recall $\omega = d\lambda$ on $\mathbb{C}$. Consider then

$$\int_{\mathbb{R}^2} \left\| \frac{\partial Y}{\partial s} \right\|^2 ds\, dt = \int_{S^2} Y^* \omega = \int_{\partial S^2} Y^* \lambda = 0.$$

Then $\frac{\partial Y}{\partial s} = \frac{\partial Y}{\partial t} = 0$. So Y is constant if it is in the kernel of $\bar{\partial}$. Hence $\ker \bar{\partial} = \mathbb{C}^n$ and has real dimension $2n$. Hence $\bar{\partial}$ is Fredholm with index $2n$. □

Using this, we prove parts of the Similarity Principle. Consider

$$D : W^{1,p}(S^2, \mathbb{C}^n) \to L^p((\Lambda^{0,1} T^* S^2)^n \oplus \mathbb{C}^n); \qquad Y \mapsto (\bar{\partial} Y, Y(0)).$$

This is the sum of two operators: $(\bar{\partial}, 0)$ and the map $Y \mapsto (0, Y(0))$. The first is Fredholm with index 0. The second is compact because inclusion of $W^{1,p}$ into L^∞ is continuous. Thus, D is Fredholm with index zero. However, $\ker D =$ (a constant $Y, Y(0) = 0$); so $\ker D = \{0\}$. Having zero index means D is surjective and hence, bijective.

Let D_δ be a small perturbation of D defined by $D_\delta(Y) = (\bar{\partial} Y + S_\delta \cdot Y d\bar{Z}, Y(0))$. $S_\delta = S$ on B_δ and rapidly decays elsewhere. Then for small enough δ, D_δ is still bijective. A Y satisfying $D_\delta Y = (0, v_0)$ satisfies the perturbed Cauchy-Riemann equations and some initial conditions. We then use such Y to build the columns of A. Through another lemma, A is $\mathbb{C}$-linear and σ is holomorphic.

5 The Fredholm Property

Let $S^\pm(t) = \lim_{s \to \pm\infty} S(s,t)$ and $R_t^\pm$ the solution of $\dot{R} = J_0 S^\pm R$ with $R_0^\pm = \mathrm{id}$.

The goal of this section is to prove the following proposition:

Proposition 5.1 (8.7.1)**.** *If $\det(\mathrm{id} - R_1^\pm) \neq 0$, then $L = \bar{\partial} + S(s,t) : W^{1,p} \to L^p$ is Fredholm, for all $p > 1$.*

We give an outline of the proof:

1. We make the assumption that $S(s,t) = S(t)$; i.e. S is independent of s. Let $D = \bar{\partial} + S(t)$; we show the stronger result that D is bijective and thus, Fredholm.

   a. We show this for $p = 2$ and take advantage of Hilbert space tools.

   b. We show this for $p > 2$. The basic idea of showing injectivity is to obtain the following inequality: $\|Y\|_{1,p} \leq C \|DY\|_p$ for all Y, some constant $C > 0$. Then, if $Y \neq 0$, $\|DY\| \neq 0$ so $DY \neq 0$. For surjectivity, show the image is dense and closed.

   c. We consider $1 < p < 2$. The adjoint $D^* : W^{1,q} \to L^q$ is defined $q > 2$; we apply the techniques from (1b) to show D^* is Fredholm, as D^* has all the relevant properties that D has for $p > 2$. If an operator is Fredholm, so is its adjoint.

2. Since $S(s,t)$ converges to something independent of s, then outside a compact set $[-C,C]\times S^1$, $L = D$. Apply the following proposition 5.2 involving compact operators. It establishes finite dimensional kernel and closed image.

3. Using the Hahn-Banach and Riesz Representation Theorems, we can prove coker $L < \infty$.

The following proposition is probably **the most important tool** in the section:

Proposition 5.2 (8.7.4)**.** *Let E, F, G be Banach spaces with $L : E \to F$ an operator, $K : E \to G$ a compact operator. Suppose there is a constant $C > 0$ s.t. $\forall x \in E$, $\|x\|_E \leq C(\|Lx\|_F + \|Kx\|_G)$. Then* $\dim\ker L < \infty$ *and the image of L is closed.*

To show that $\dim \text{coker } L < \infty$, we identify coker L with $\ker L^*$: since Im $L \perp \ker L^*$, $L^p = \ker L^* \oplus \text{Im } L$. So then $\ker L^* = W/\text{Im } L = \text{coker } L$. Now, we need only show $\dim\ker L^* < \infty$ but since we have the previous proposition, once we show L^* satisfies the hypothesis, we can achieve this. This discussion also applies to D.

Let us omit the proof that D is continuous and bijective and just assume these results. The inverse of D is continuous by the Open Mapping Theorem. Then, there is a $B > 0$ s.t. $\|Y\|_{1,p} = \|D^{-1}DY\|_{1,p} \leq B\|DY\|_p$.

Since $S \to S^{\pm}$ as $s \to \pm\infty$, there are constants $M, C < 0$ such that if $Y(s,t) = 0$ when $|s| \leq M-1$, then $\|Y\|_{1,p} \leq C\|LY\|_p$. This is because, outside of $[-M, M]$, $\|S - S^{\pm}\|$ is small; one can increase M if needed. Then, outside this compact set, $LY = DY$.

Let $\beta : \mathbb{R} \to [0,1]$ be a bump function which is 1 on $[1-M, M-1]$ and 0 on $\mathbb{R} - [-M, M]$. Write $Y = \beta Y + (1-\beta)Y$. The derivative $|\beta'(s)|$ is bounded so we're able to show $\|L(\beta Y)\|_p \leq \|LY\|_p + K\|Y\|_{L^p[-M,M]}$ for some $K > 0$. Finally, we obtain the inequality with some $C_2 > 0$

$$\|Y\|_{1,p} \leq C_2(\|Y\||_{L^p[-M,M]} + \|LY\|_p).$$

We're almost in position to use Prop 5.2. We just need to confirm that we have a compact operator:

Theorem 5.3 (Rellich's Theorem; Appendix C.4.6)**.** *Let $U \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain (the boundary is the graph of a Lipschitz function). Let $p > n$. Then $W^{1,p}(U; \mathbb{R}^m)$ is a subspace of $C^0(U, \mathbb{R}^m)$ and the injection $W^{1,p} \hookrightarrow C^0$ is a compact operator.*

Rellich's theorem implies that the restriction operator to $[-M, M] \times S^1$ is a compact operator. So we can apply Proposition 5.2. Thus, L has finite dimensional kernel and closed image. The last thing to show is that the cokernel is finite dimensional. The main step is to identify coker $L = \ker L^*$.

Let $L^* : W^{1,q} \to L^q$ be the adjoint of L $(1/p + 1/q = 1)$ and $F \subset L^q$ be the subspace of vector fields Z orthogonal to the image of L: $\langle \text{Im } L, Z\rangle = 0$. By elliptic regularity, since $L^*Z = 0$, $Z \in W^{1,q}$. Thus, $F \subset \ker L^*$. L^* satisfies the same conditions as L so we can conclude that $\dim\ker L^* < \infty$.

Now, the Hahn-Banach theorem allows us to find linear forms $\varphi : L^p \to \mathbb{R}$ that are zero on Im L. We want to show the space of these forms is finite-dimensional as that will show coker L is finite-dimensional. The Riesz Representation Theorem allows us to write a linear form with a representative $U \in L^q$, as $\varphi(V) = \langle U, V\rangle$. Since φ vanishes on Im L, then $U \in F$. But $\dim F < \infty$ so the space of these forms is finite-dimensional and consequently, coker $L < \infty$.

6 Computing the Index of L

It may be more enlightening to refer to D. Salamon's *Lectures on Floer Homology* and see how to use spectral flow to compute the index. However, Audin and Damian also have a computation. Recall that $L = \bar{\partial} + S(s,t)$ where $S(s,t) \to S^{\pm}(t)$ as $s \to \pm\infty$, uniformly in t.

To compute the index, we:

1. Replace L by L_0 which has the same formula except we replace S by a matrix $\widetilde{S}$ which is *exactly* S^- for some $s \leq -\sigma_0$ and *exactly* S^+ for some $s \geq \sigma_0$. The index is invariant under small perturbations for sufficiently large σ_0. So L_0 and L will have the same index.

2. Replace L_0 by L_1 which has the same formula except $\widetilde{S}$ is replaced by a *diagonal* matrix $S(s)$ that is independent of t and is constant for $|s| > \sigma_0$. L_1 will have the same index as L_0 because of the invariance of index under homotopy. We are able to compute the index because we can describe the kernel and cokernel of L_1. Of course, there is no reason for the dimensions to be invariant. It is the index which is invariant.

7 Exponential Decay

Recall the definition of $C^\infty_\searrow(x, y)$. It consists of maps $u : \mathbb{R} \times S^1 \to M$ such that u limits to periodic orbits x and y and there exist $K, \delta > 0$ such that

$$\left|\frac{\partial u}{\partial s}(s,t)\right| \leq K e^{-\delta|s|} \qquad \left|\frac{\partial u}{\partial t}(s,t) - H_t(u)\right| \leq K e^{-\delta|s|}.$$

Proposition 7.1 (8.2.3)**.** *If x and y are contractible loops and nondegenerate critical points of $\mathcal{A}_H$, then $\mathcal{M}(x, y) \subset C^\infty_\searrow(x, y)$.*

The proof of this proposition relies on proving:

Theorem 7.2 (8.9.1)**.** *If Y is a C^2 solution of the Floer equation linearized at a finite energy solution, then:*

- *Either $\int \|Y\|^2\, dt$ tends to $+\infty$ when $s \to \pm\infty$*
- *Or Y satisfies $\|Y(s,t)\| \leq Ce^{-\delta|s|}$ for certain constants δ and C and for every t.*

The idea of the proof begins by defining a C^2 function $f : \mathbb{R} \to \mathbb{R}$, $f(s) = \frac{1}{2}\|Y\|^2_{L^2(S^1)}$. We show that $f'' \geq \delta^2 f$ for some constant δ, and then show that such functions must satisfy an analogous exponential decay theorem as above. This will guarantee that for all $s \in \mathbb{R}$ and fixed t, $\|Y\|^2_{L^2(S^1)} \leq e^{-\delta|s|}$. However, we are not yet there as we want to show that for all $(s,t) \in \mathbb{R} \times S^1$, $\|Y(s,t)\| \leq Ce^{-\delta|s|}$.

We'll need two results:

Lemma 7.3 (8.9.5)**.** *Let Y be a C^2 solution of the linearized Floer equation. There exists a constant $a > 1$ such that $\Delta\|Y\|^2 \geq -a\|Y\|^2$.*

Proposition 7.4 (8.9.6)**.** *Let $w : \mathbb{R}^2 \to \mathbb{R}$ be a positive C^2 function such that $\Delta w \geq -aw$ for a constant $a > 1$. We then have*

$$\forall (s_0, t_0) \in \mathbb{R}^2, \quad w(s_0, t_0) \leq \frac{8a}{\pi} \int_{B_1(s_0,t_0)} w(s,t)\, ds\, dt.$$

Remark: This is something like a Mean Value Inequality for maps that satisfy a harmonic relation.

When we let $w = \|Y(s,t)\|^2$, then the proposition says that

$$\|Y(s_0,t_0)\|^2 \leq C \int_{B_1(s_0,t_0)} \|Y(s,t)\|^2\, ds\, dt = C\|Y\|^2_{L^2(B_1(s_0,t_0))}$$

On the other hand, we know that

$$\|Y\|^2_{L^2(S^1)} = \int_0^1 \|Y(s,t)\|^2 \, dt \le e^{-\delta|s|}$$

Note that for a fixed s_0, the circle going through s_0 is contained in $B_1(s_0,t_0)$ for any $t_0 \in S^1$. If we consider the square $Q = [s_0-1, s_0+1] \times [t_0-1, t_0+1]$ centered at (s_0,t_0), we now have an upper bound

$$\|Y\|^2_{L^2(B_1(s_0,t_0))} \le \|Y\|^2_{L^2(Q)} = \int_{s_0-1}^{s_0+1} f(s) \, ds \le \int_{s_0-1}^{s_0+1} e^{-\delta|s|} \, ds$$

Say $s_0 \ge 1$. Then the last integral equals

$$-\frac{1}{\delta}(e^{-\delta(s_0+1)}) - e^{-\delta(s_0-1)}) = \frac{(e-e^{-1})}{\delta} e^{-\delta s_0} = Ce^{-\delta s_0},$$

C being the constant. The other cases are similar. As we change (s_0,t_0), the upper bound changes like $e^{-\delta|s|}$.