Outline of Ch. 7: The Conley-Zehnder Index

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Let J_0 be the standard almost complex structure on $\mathbb{R}^{2n}$. Denote $Sp(2n) = \{A \in GL(n, \mathbb{R}) : A^t J_0 A = J_0\}$ as the group of symplectic $2n \times 2n$ matrices. Conley and Zehnder introduced a way to assign an index for paths of symplectic matrices. Consider a path $\Psi : [0, 1] \to Sp(2n)$ such that $\Psi(0) = \text{id}$ and $\det(\text{id} - \Psi(1)) \neq 0$.

Let $Sp^*(2n)$ be symplectic matrices without 1 as an eigenvalue. This set is open and dense in $Sp(2n)$ and has two components; its complement is called the **Maslov cycle** which forms a codimension 1 algebraic variety with natural co-orientation defined by the equation $\det(\text{id} - A) = 0$. Thus, we may split $Sp^*(2n)$ naturally into a "positive" and "negative" part.

Now, consider ρ, a continuous extension of $\det : U(n) = Sp(2n) \cap O(2n) \to S^1$. ρ is not a group morphism but it can be chosen to be multiplicative with respect to direct sums, invariant under similarity, and taking the value ± 1 for symplectic matrices with no eigenvalues on S^1. These properties uniquely determine ρ.

Now let $SP(2n)$ be the space of paths $\Psi : [0, 1] \to Sp(2n)$ with $\Psi(0) = \text{id}$ and $\Psi(1) \in Sp^*(2n)$. Any such path admits an extension $\Psi : [0, 2] \to Sp(2n)$, unique up to homotopy, such that $\Psi(s) \in Sp^*(2n)$ for $s \geq 1$ and $\Psi(2)$ is one of the following matrices: $W^+ = -\text{id}$ or $W^- = \text{diag}(2, -1, ..., -1, 1/2, -1, ..., -1)$. Since $\rho(W^\pm) = \pm 1$, it follows that $\rho^2(W^\pm) = +1$ and so $\rho^2 \circ \Psi : [0, 2] \to S^1$ is a loop. The **Conley-Zehnder index** of Ψ is defined as the degree: $\mu_{CZ}(\Psi) = \deg \rho^2 \circ \Psi$. This counts the number of **counterclockwise** half turns around S^1. This is almost the Maslov index that Audin and Damian define in their book but with opposite sign; Audin and damian count **clockwise** half turns.

So how do we assign an index to a periodic orbit $x : S^1 \to (M, \omega)$? Since we're looking at contractible loops, we can extend x to $\bar{x} : D^2 \to M$; it is a result that over D^2, all symplectic bundles can be trivialized and the trivializations are all homotopic. This means we can choose a symplectic trivialization $\{Z_i\}^{2n}$ of $\bar{x}^* TM$ and consider it restricted to x.

The assumption that $\langle c_1, \pi_2(M) \rangle = 0$ means that letting $c_1 = c_1(M)$, for all smooth maps $f : S^2 \to M$,

$$\int_{S^2} f^* c_1 = 0.$$

This ensures that our choice of extension $\bar{x}$ does not matter in the end. Suppose u and v are two extensions of x; we glue them to form f. Then,

$$0 = \int_{S^2} f^* c_1 = \int_{D^2} u^* c_1 - \int_{D^2} v^* c_1.$$

Another view of this is via the **clutching construction**; gluing two capping discs along an S^1 to form an $\mathbb{C}P^1 = S^2$ means we need a gluing map on S^1. This is asking about transition

(gluing) maps on the two charts of $\mathbb{C}P^1$; the transition map are classified by $\pi_1 Sp(2n) = \pi_1 U(n) = \mathbb{Z}$. Let $f : S^2 \to M$ be a map constructed by gluing two extensions of x together. Fixing a basepoint, f is determined by some gluing map $g \in \pi_1 Sp(2n)$ and $[f] \in \pi_2 M$. If $\pi_2 M = 0$, we see immediately that $[f]$ is contractible and the two disks are homotopic (can slide through B^3) and so we'll obtain homotopic paths in $Sp(2n)$.
Question: Why should we expect c_1 to be involved? Since $Sp(2n)$ deformation retracts to $U(n)$, we know that $\mathbb{Z} = \pi_1 Sp(2n) = \pi_2 BSp(2n)$. Since $Sp(2n)$ is connected, $\pi_1 BSp(2n) = 0$. By Hurewicz's theorem, $c_1 \in H^2(BSp(2n), \mathbb{Z}) = \pi_2 BSp(2n) = \mathbb{Z}$. Our assumption is that $0 = c_1(f^*TM) \in H^2(S^2; \mathbb{Z}) = \mathbb{Z}$. This means that our gluing map $[g] = 0 \in \pi_1 Sp(2n)$. In this case, the trivializations can slide from one disk to the other through B^3; i.e. are homotopic.

Now, let $\Psi : [0, 1] \to Sp(2n)$ be the path sending $t \mapsto A(t)$ where $A(t)$ is the matrix for $(d\varphi_t)_{x_0}$ in the trivialization Z_i. Because x is nondegenerate, $\Psi(1)$ does not have eigenvalue 1. We can now apply the above concepts to define the Maslov index for x.

This index has the following properties. It is uniquely determined by the homotopy, loop, and signature properties.

1. **(Naturality)** For any path $\Phi : [0, 1] \to Sp(2n)$, $\mu_{CZ}(\Phi\Psi\Phi^{-1}) = \mu_{CZ}(\Psi)$.
2. **(Homotopy)** The index is constant on the components of $SP(2n)$.
3. **(Zero)** If $\Psi(s)$ has no eigenvalue on the unit circle for $s > 0$, then $\mu_{CZ}(\Psi) = 0$.
4. **(Product)** If $n' + n'' = n$, identify $Sp(2n') \oplus Sp(2n'')$ with a subgroup of $Sp(2n)$ in the obvious way. Then $\mu_{CZ}(\Psi' \oplus \Psi'') = \mu_{CZ}(\Psi') + \mu_{CZ}(\Psi'')$.
5. **(Loop)** If $\Phi : [0, 1] \to Sp(2n, \mathbb{R})$ is a loop with $\Phi(0) = \Phi(1) = \mathrm{id}$, then $\mu_{CZ}(\Phi\Psi) = \mu_{CZ}(\Psi) + 2\mu(\Phi)$.
6. **(Signature)** If S is a nondegenerate symmetric matrix with $\|S\| < 2\pi$ and $\Psi(t) = \exp(tJ_0S)$, then $\mu_{CZ}(\Psi) = \frac{1}{2}\sigma(S)$ where $\sigma(S)$ is the signature (# positive eigenvalues − # negative eigenvalues).
7. **(Determinant)** $(-1)^{n-\mu_{CZ}(\Psi)} = \operatorname{sign}\det(\mathrm{id} - \Psi(1))$.
8. **(Inverse)** $\mu_{CZ}(\Psi^{-1}) = \mu_{CZ}(\Psi^t) = -\mu_{CZ}(\Psi)$.

Observe that if S is a nondegenerate symmetric matrix with $\|S\| < 2\pi$ and we let $\Psi(t) = \exp(tJ_0S)$, then we can express (6) in a different way. Let k = # of negative eigenvalues of S. Then, being nondegenerate, S has $2n - k$ positive eigenvalues. (6) tells us $\mu_{CZ}(\Psi) = \frac{1}{2}(2n - k - k) = n - k$. This is opposite to what Audin and Damian have which is $\mu(\Psi) = k - n$. Similarly, in Audin and Damian, (7) is given as $(-1)^{\mu(\Psi)-n} = \operatorname{sign}\det(\mathrm{id} - \Psi(1))$.