Outline of Ch. 10: Floer and Morse Theory Coincide for Autonomous, “Small” Hamiltonians

Sam Auyeung

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We give an outline of ch. 10 of *Morse Theory and Floer Homology.* The goal of chapter 10 is to show that in the case of a nondegenerate autonomous Hamiltonian which is sufficiently small in the C^2 topology, if we can define a Morse and Floer complex, the two coincide. Before we consider the case of time-dependent Hamiltonians, we discuss autonomous, C^2 small Hamiltonians.

0 Proof of Arnold's Conjecture for autonomous C^2 small Hamiltonians

The following if a proposition from ch. 5. It says that if a critical point of an autonomous Hamiltonian H is nondegenerate in the Floer theory sense, it is also nondegenerate in the Morse theory sense.

Proposition 0.1 (5.4.5)**.** *If a critical point of H is nondegenerate as a periodic solution of the Hamiltonian system, then it is nondegenerate as a critical point of the function H.*

Proof. Let x be a critical point of H. First, we show that $(d^2H)_x(Y,Z) = \omega_x([X_H, Z], Y)$. To do this, extend the vector Y at x to a Hamiltonian vector field X_f in a neighborhood. Then in this neighborhood at x (we'll drop the x eventually),

$$\begin{aligned}\omega_x([X_H, Z], Y) &= \omega_x([X_H, Z], X_f)\\ &= df_x([X_H, Z])\\ &= [X_H, Z]\cdot f\\ &= X_H\cdot(Z\cdot f) - Z\cdot(X_H\cdot f)\end{aligned}$$

In the second term, we have $X_H\cdot f = df(X_H) = \omega(X_H, X_f) = -\omega(X_f, X_H) = -dH(X_f) = -X_f\cdot H$. Also, the first term is 0 since x is a critical point: $X_H(x) = 0$ since $\omega_x(X_H, Y) = -dH_x(Y) = 0, \forall Y$. As ω is nondegenerate, $X_H(x) = 0$. Thus, the last line gives us that $\omega_x([X_H, Z], Y) = Z\cdot(X_f\cdot H) = Z\cdot dH_x(X_f) = (d^2H)_x(X_f(x), Z) = (d^2H)_x(Y, Z)$. This proves the first step.

Next, since we're supposing x to be nondegenerate as a periodic orbit, $(d\psi^1)x$ does not have eigenvalue 1. Thus, for all $Z \neq 0$, $(d\psi^1)x(Z) - Z \neq 0$. However, $\psi^0 = \text{id}$ so $(d\psi^0)x(Z) - Z = 0$. This means, that $(d\psi^t)x(Z) - Z$ must go from 0 to something nonzero as t changes; i.e. there is a $t \in (0,1)$ such that $\frac{d}{dt}(d\psi^t)_x(Z) \neq 0$. But

$$\frac{d}{dt}(d\psi^t)_x(Z) = \lim_{h\to 0}\frac{(d\psi^{t+h})_x(Z) - (d\psi^t)_x(Z)}{h} = (d\psi^t)_x\lim_{h\to 0}\frac{(d\psi^h)_x(Z) - Z}{h}.$$

The last limit is the Lie derivative of Z with respect to X_H (the ψ^t is the flow of X_H). But $\mathcal{L}_{X_H} Z = [X_H, Z]$. So there is a t such that $(d\psi^t)_x([X_H, Z]) \neq 0$ which means $[X_H, Z] \neq 0$. Lastly, for any fixed Z and every Y, $(d^2H)_x(Y, Z) = \omega_x([X_H, Z], Y)$. The nondegeneracy of the RHS assures the nondegeneracy of the LHS. So x is nondegenerate in the Morse theory sense. $\square$

The following two propositions are from ch. 6. They show, in two steps, that periodic orbits for autonomous C^2 small Hamiltonians are constant. If we further assume H is nondegenerate, then these critical points coincide with the Morse critical points; i.e. $\mathrm{Crit}(\mathcal{A}_H) = \mathrm{Crit}(H)$. Thus, H is a Morse function.

Proposition 0.2 (6.1.5)**.** *Let H be a function on $\mathbb{R}^{2n}$ so that X_H is a vector field on $\mathbb{R}^{2n}$. If $\|dX_H\|_{L^2} < 2\pi$, then the only solutions of period 1 of the Hamiltonian system associated with H are the constant solutions (critical points of H).*

Proof. Consider a solution x of period 1 and take its Fourier expansion (it's a vector in $\mathbb{C}^n$) as well as those of its derivatives $\dot{x}$, $\ddot{x}$:

$$x(t) = \sum_n c_n(x) e^{2\pi i n t}; \qquad \dot{x}(t) = \sum_n 2ni\pi c_n(x) e^{2\pi i n t}; \qquad \ddot{x}(t) = -\sum_n 4\pi^2 n^2 c_n(x) e^{2\pi i n t}$$

Since $c_n(\dot{x}) = 2\pi i n c_n(x)$ and $c_0(\dot{x}) = 0$, Parseval's Identity gives us

$$\|\ddot{x}\|^2_{L^2} = 4\pi^2 \sum n^2 |c_n(\dot{x})|^2 \geq 4\pi^2 \sum |c_n(\dot{x})|^2 = 4\pi^2 \|\dot{x}\|^2_{L^2}.$$

Therefore, $2\pi\|\dot{x}\|_{L^2} \leq \|\ddot{x}\|_{L^2}$. Since $\ddot{x} = (dX_H)_x \cdot \dot{x}$, the condition $\|dX_H\| < 2\pi$ gives that $\|\ddot{x}\|_2 \leq \|dX_H\|_2 \|\dot{x}\|_2 < 2\pi\|\dot{x}\|_2$ if $\dot{x} \neq 0$. So then combining these, $\|\dot{x}\|_2 < \|\dot{x}\|_2$ which is impossible so it must be that $\dot{x} = 0$ and x is constant. $\square$

Observe that if H is time dependent, then $\ddot{x} = (dX_H)_x \cdot \dot{x} +$ extra terms depending on $\frac{d}{dt} H_t$. Unless we have some conditions on the bounds of these derivatives, the proof can't follow through.

However, Floer homology is an invariant that doesn't depend on the type of Hamiltonian. We may homotope any arbitrary Hamiltonian to one that is both autonomous and C^2 small.

Proposition 0.3 (6.1.6)**.** *Let (M, ω) be a compact symplectic manifold and let $H : M \to \mathbb{R}$ be a function. If H is sufficiently C^2 small, then the only solutions of period 1 of the Hamiltonian system are the constant solutions.*

Proof. If we look at a Hamiltonian vector field X_H on a disk $D^{2n} \in \mathbb{R}^{2n}$, then $\forall x \in D$, $\forall t \in [0, 1]$, $\|\varphi_t(x) - x\| \leq \sup_{y \in D} \|X_H(y)\|$.

The way to prove this claim is by the following: let us fix an $x \in D$ and consider $f(t) := \|\varphi_t(x) - x\|$. Note that $f(0) = 0$. By the Mean Value Theorem, there is an s such that

$$\frac{|f(t) - f(0)|}{|t - 0|} = \frac{f(t)}{t} = f'(s).$$

Let's compute $f'(s)$. In terms of its component functions,

$$f(t) = \sqrt{\sum_i^{2n} (\varphi_t^i(x) - x)^2}$$

so then by Chain Rule,

$$\frac{d}{dt}f(t)\bigg|_{t=s} = \frac{1}{2\|\cdot\|}\sum^{2n} 2(\varphi_s^i(x) - x)\cdot\dot{\varphi}_s^i(x).$$

Applying the Cauchy-Schwarz inequality, we have that the derivative above has norm less than

$$\frac{\|\cdot\|}{\|\cdot\|}\sqrt{\sum^{2n}(\dot{\varphi}_s^i)^2} = \|X_H\| = \sup_{y\in D}\|X_H(y)\|.$$

Putting this all together and the fact that $t \leq 1$, we have

$$f(t) \leq t\sup_{y\in D}\|X_H(y)\| \implies \|\varphi_t(x) - x\| \leq \sup_{y\in D}\|X_H(y)\|.$$

In particular, if we consider a Hamiltonian orbit, because the above holds for $t \in [0,1]$, the orbit cannot be very large; it's bounded by $\sup_{y\in D}\|X_H(y)\|$. Also, M is compact so we can cover M by a finite number of relatively compact Darboux charts.

Thus, if $H : M \to \mathbb{R}$ is sufficiently C^2 small, then this places a bound on $\|dX_H\|$ which places a bound on $\|X_H\|$. We can make $\|dX_H\|$ small enough (in particular, smaller than 2π) that the period 1 orbits are forced into a Darboux chart and we can apply the previous proposition to show that the orbits must in fact be constants. □

1 The Theorems and an Outline

Let J be an almost complex structure on symplectic manifold (W,ω) and $CF_*(H,J)$ denote the Floer complex associated to a Hamiltonian H and J. Let $CM_*(H,J)$ denote the Morse complex associated to the Morse function H and the vector field ∇H with respect to the metric defined by ω and J.

Theorem 1.1 (Main Theorem)**.** *[10.1.1] There exists a nondegenerate, sufficiently small (in the C^2 sense) Hamiltonian H for which $CF_*(H,J) = CM_{*+n}(H,J)$.*

This theorem is saying that the complexes coincide and that there is an H such that both the complexes are well-defined. Note that the Maslov index can be negative and for C^2 small Hamiltonians, the indices fall inside $[-n,n]$ where $\dim M = 2n$. Thus, the shift in indices between the two complexes is no issue. Also, to reiterate section 0, when H is C^2 small, its periodic trajectories are constants; hence $\mathrm{Crit}(\mathcal{A}_H) = \mathrm{Crit}(H)$ (prop. 6.1.5). On the other hand, prop. 5.4.5 shows that if a critical point of H is nondegenerate as a periodic trajectory, it is nondegenerate as a critical point. This implies that H is a Morse function.

If x is a critical point of H, then $\mathrm{Hess}_x H$ has no eigenvalues in $2\pi\mathbb{Z}$. Then prop 7.2.1 allows us to compare the Morse and Maslov indices of x as a critical point and x viewed as a constant periodic orbit: $\mathrm{Ind}_H(x) = \mu(x) + n$.

To define the differential of the Morse complex, we need a Smale pseudo-gradient field X adapted to the Morse function H. But we also want a relationship between the trajectories of X and solutions of the Floer equation; i.e. we want a relationship between solutions to

$$\frac{du}{ds} + X(u) = 0 \quad \text{and} \quad \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(u) = 0.$$

It's natural to let $X = \nabla H$; then the PDE's are closely related. Since H is small, we'll show that a solution u has $\partial u/\partial t = 0$. Here are some theorems we'll prove along the way:

Theorem 1.2 (10.1.2)**.** *Let H be a Morse function on symplectic manifold (W, ω). There is a dense subset, call it $\mathcal{J}_{reg}(H)$, consisting of almost complex structures J calibrated by ω such that the pair $(H, -JX_H)$ is Morse-Smale.*

To prove this theorem, we take two steps. First, look at a general Morse function f with adapted vector field X. We linearize the equation $du/ds + X(u) = 0$ of the flow of $-X$ along a solution u. We obtain from this, an operator L_u and we're looking at the equation $L_u Y = 0$. We then prove the following result:

Theorem 1.3 (10.1.3)**.** *When f is Morse and u is a trajectory connecting critical points x and y, L_u is Fredholm with index equal to $Ind(x) - Ind(y)$ (we're taking Morse index).*

Corollary 1.4. *For nondegenerate autonomous H and trajectories u of $-JX_H$, the Fredholm operators $(d\mathcal{F})_u$ and L_u have the same index.*

The proof is simply using that $\mathrm{Ind}(x) - \mathrm{Ind}(y) = \mu(x) + n - \mu(y) - n$. We also show the following:

Theorem 1.5 (10.1.5)**.** *X is Smale if and only if the operators L_u are surjective.*

The second step is to prove a transversality result similar to in ch. 8. Next, we prove a proposition:

Proposition 1.6 (10.1.7)**.** *If H is sufficiently small, then* $\ker(d\mathcal{F})_u = \ker L_u$.

Since the time independent solutions to the Floer equation are precisely the trajectories of JX_H, this proposition tells us that the elements in the kernel of $(d\mathcal{F})_u$ are independent of t. Then, since JX_H is Smale by the regularity thm 10.1.2, L_u is surjective. L_u and $(d\mathcal{F})_u$ have the same kernel and index. Thus, $(d\mathcal{F})_u$ must be surjective as well. Moreover, if we let $H_k = H/k$, then we get a result:

Proposition 1.7 (10.1.9)**.** *If k is sufficiently large, the solutions of the Floer equation for H_k connecting critical points x and y with $Ind_{H_k}(x) - Ind_{H_k}(y) \leq 2$ are all independent of t.*

The conclusion is thus: Let H_k be our Hamiltonian of interest for large enough k and let $J \in \mathcal{J}_{\mathrm{reg}}$. When $\mathrm{Ind}(x) - \mathrm{Ind}(y) \leq 2$, trajectories of the Floer equation associated with (H_k, J) that connect critical points x and y are exactly the trajectories of the Smale vector field $X = -JX_H$. The Floer operator linearized along these trajectories is surjective. Such a regularity condition implies that $\mathcal{M}(x, y, J, H_k)$ is a manifold. We can therefore, define the Floer complex. From our discussions, the Morse and Floer complexes do indeed coincide.

2 Linearization of the Flow of a Pseudo-Gradient Field

Let V be a smooth manifold with Morse function f; embed V into some $(\mathbb{R}^m, g)$ with a metric. Consider the space $\mathcal{M} = \{u : \mathbb{R} \to V : du/ds + \nabla f = 0, \int_{\mathbb{R}} |du/ds|^2 < \infty\}$. These are trajectories with finite energy. When V is compact, all trajectories have finite energy. This can be seen in the following way: define energy by

$$E(u) = -\int_{-\infty}^{\infty} u^* df = -\int_{-\infty}^{\infty} df(du/ds) = -\int_{-\infty}^{\infty} df(-\nabla f) = +\int_{-\infty}^{\infty} \|\nabla f\|^2 = \int_{-\infty}^{\infty} |du/ds|^2.$$

On a compact manifold V, the finite energy trajectories must connect critical points; if they don't, the trajectory just keeps going and thus, can't have finite energy. Similarly, even on a non-compact manifold V, the trajectories with finite energy must connect critical points of f. Now, $\frac{d}{ds} f(u(s)) = df \circ du/ds = u^* df$. Thus, if $\lim_{s \to -\infty} u(s) = a$ and $\lim_{s \to +\infty} u(s) = b$, then

$$E(u) = -\int_{-\infty}^{\infty} u^* df = -\int_{-\infty}^{\infty} \frac{d}{ds} f(u(s))\, ds = f(a) - f(b) < \infty.$$

Claim: $\mathcal{M}$ is compact when V is compact.

Proof. This is proven by Arzelà-Ascoli. In general, on a compact space, if we have a sequence of functions, then if they are equicontinuous and uniformly bounded, there exists a subsequence which converges. However, if we're looking at differentiable functions, if the derivatives of the functions are uniformly bounded, then they converge. Well, the derivative of a solution u is $du/dt = -\nabla f$ and so they are all bounded by $\|\nabla f\|$. Thus, by Arzelà-Ascoli, $\mathcal{M}$ is compact. $\square$

2.1 Linearization of the Equation

The upshot is that when we use an embedding into some large $\mathbb{R}^n$ and properly chosen trivializations, we obtain an operator

$$L_u : W^{1,2}(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n), \qquad Y \mapsto \frac{dY}{ds} + A(s)Y.$$

This A is some operator and $\lim_{s \to -\infty} A(s) = \mathrm{Hess}_x(f)$ and $\lim_{s \to +\infty} A(s) = \mathrm{Hess}_y(f)$; these are symmetric matrices. Notice the 2; since the dimension of the domain is 1, $W^{1,2} \subset C^0$.

2.2 The Exponential Decay of the Solutions

If our trivialization is given as (Z_i), then we may write a $W^{1,2}$ section $Y = \sum y_i Z_i$. Then, at $\pm\infty$, we have that A is converging to the Hessians which are nondegenerate and we may assume to be diagonal under our trivializations. Say, when $s \to -\infty$, $A(s) = \mathrm{diag}(\lambda_i)$ and when $s \to +\infty$, $A(s) = \mathrm{diag}(\mu_i)$. We'll just assume $s \to -\infty$ as the $+\infty$ case is similar. Then

$$\frac{dY}{ds} = -AY \Longrightarrow \frac{dy_i}{ds} = -\lambda_i y_i \Longrightarrow y_i(s) = y_i(0) e^{-\lambda_i s}.$$

For Y to be $W^{1,2}$, it decays exponentially at $\pm\infty$. In more detail,

$$\|Y\|_{1,2}^2 = \int_{-\infty}^{\infty} |Y|^2 + |dY/ds|^2\, ds$$

Therefore, as $s \to -\infty$, we need the integrand to go to zero. This means that the $\lambda_i < 0$. It appears that this is a contradiction because as $s \to -\infty$, A approaches $\mathrm{Hess}_x f$, which may have **positive** eigenvalues. However, we must remember that we linearized along a solution u and we can only go backwards, along u, towards x in the unstable manifold $W^u(x)$. Hence, A converges towards $\mathrm{diag}(\lambda_i)$ and will always be in the negative eigenspace of $\mathrm{Hess}_x f$.

2.3 The Fredholm Property

We wish to show that L_u is Fredholm. The proof is quite similar to the Floer theory case (see chapter 8). In particular, we make use of prop. 8.7.4 which deals with an operator and compact operator together giving some bound. If the hypothesis holds, we automatically find that L_u has finite kernel and closed image. To obtain the compact operator, we split the domain into two pieces: $[-T, T]$ and $\mathbb{R} - [-T, T]$. In this way, we're able to apply the proposition. To show that the cokernel is finite dimensional, we consider L_u^* and show its kernel is finite dimensional. A lot of this theory only depends on $W^{1,2}$ and thus, L^2, a self-dual Hilbert space.

2.4 Index of L_u

Let $\dim M = 2n$. The authors prove that for an autonomous C^2 small Hamiltonian H, the Maslov index and Morse index of a critical point x of H have the following relation: $\text{Ind}_H(x) = \mu(x)+n$. Then, they show that L_u has index $\text{Ind}(x)-\text{Ind}(y) = \mu(x)+n-(\mu(y)+n) = \mu(x)-\mu(y)$ which is the index of $(d\mathcal{F})_u$. This is done by considering the dimensions of stable and unstable manifolds. Thus, Thm 10.1.3 and its corollary are proved. Thm 10.1.5 is also proved in the course of the computation of the index.

3 Regularity: Perturbations of the Almost Complex Structure

The goal of this section is to prove theorem 10.1.2 (see the introduction above). We begin by considering a fixed Morse function H and consider two of its critical points x, y. Let $Z(x,y,H) = \{(u,J)\}$ where J is an ω-compatible ACS and u is a trajectory of $-JX_H$ connecting x, y. Now, if we consider the space of ω-compatible ACS $\mathcal{J}_c(\omega)$, it can be given smooth structure since $\mathcal{J}_c(\omega)$ is a subspace of smooth sections for the bundle $\text{End}(TW)$. Then, the tangent space at J is

$$T_J\mathcal{J}_c(\omega) = \{S \in \text{End}(TW) : JS + SJ = 0,\ \omega(Sx,y) + \omega(x,Sy) = 0\}.$$

Just to justify this, note that if we have a path $J(t) \in \mathcal{J}_c(\omega)$ (and call $J := J(0)$), then $J(t)^2 = -\text{id}$. Differentiating both sides, we get $J'(t)J(t) + J(t)J'(t) = 0$. Letting $t = 0$ and calling $S = J'(0)$, we have $SJ+JS = 0$. As for the second equation, the compatibility condition gives us a family of Riemannian metrics $g_t(x,y) = \omega(x, J(t)y)$. They are symmetric so we have that $0 = g_t(x,y) - g_t(y,x) = \omega(x, J(t)y) - \omega(y, J(t)x)$. Differentiating and letting $t = 0$, we have $\omega(x, Sy) + \omega(Sx, y) = 0$.

Now, fix such an S and let $J_t = J\exp(-tJS)$. With some linear algebra, we can prove that $J_t \in \mathcal{J}_c(\omega)$ for small t.

Recall that C^∞_ϵ is a space of perturbations of some object and has a specially defined norm. In ch. 8, we perturbed the Hamiltonians. Here, we'll perturb an ACS. Let $\mathcal{J}_0(\delta) = \{J_0\exp(-J_0S) : S \in C^\infty_\epsilon(J_0), \|S\|_\epsilon < \delta\}$. Now let $Z_0(x,y)$ be a subset of $Z(x,y,H)$ consisting of pairs (u,J) where $J \in \mathcal{J}_0(\delta)$.

Proposition 3.1 (10.3.3)**.** *$Z_0(x,y)$ is a Banach manifold.*

Once this is established, we conclude the proof of theorem 10.1.2 by considering the projection $\pi : Z_0(x,y) \to \mathcal{J}_0(\delta)$, $(u,J) \mapsto J$. π is Fredholm and the Sard-Smale theorem gives us a countable intersection of dense open sets in $\mathcal{J}_0(\delta)$, consisting of regular values. Call this set of regular values $\mathcal{J}_{\text{reg}}(x,y)$. Note that this is dependent on the critical points x, y. However, we can apply Sard-Smale to all pairs of critical points and thus, take an intersection of all the $\mathcal{J}_{\text{reg}}(x,y)$ to obtain a dense set of regular values. We state this result as an important lemma:

Lemma 3.2 (10.3.6)**.** *If $J \in \mathcal{J}_{reg}(x,y)$, then for every trajectory u of the vector field $-JX_H$ connecting x and y, the linearized operator L_u is surjective.*

4 Morse and Floer Trajectories Coincide

The goal of this section is to prove prop. 10.1.7. Let's first focus on trajectories that **do not depend** on t. Then it is clear that $\ker L_u \subset \ker(d\mathcal{F})_u$. To get the opposite inclusion, suppose

$Y \in \ker(d\mathcal{F})_u$ and Y is independent of t. Then

$$\int_0^1 J\frac{\partial Y}{\partial t}\,dt = 0 \Longrightarrow \widehat{Y}(s) := \int_0^1 Y(s,t)\,dt \in \ker L_u.$$

To show the last claim that $\widehat{Y} \in \ker L_u$, just consider how

$$L_u\widehat{Y} = \frac{d\widehat{Y}}{ds} + A(s)\widehat{Y} = \int_0^1 \frac{\partial Y}{\partial s}\,dt + A(s)\widehat{Y}.$$

Since Y is a solution of the linearized Floer equation, this means $\frac{\partial Y}{\partial s} = -J\frac{\partial Y}{\partial t} - A(s)Y$. Substituting, we find

$$L_u\widehat{Y} = -\int_0^1 J\frac{\partial Y}{\partial t}\,dt - \int_0^1 A(s)Y\,dt + A(s)\int_0^1 Y\,dt = 0.$$

This last equality holds because Y is independent of t and so is $A(s)$; i.e. we can move $A(s)$ inside the integral. Through a lemma, we can show that in fact, $Y - \widehat{Y} = 0$ and so $Y = \widehat{Y} \in \ker L_u$. Thus, $\ker L_u = \ker(d\mathcal{F})_u$ for trajectories that are independent of t.

Next, we show that all trajectories for autonomous C^2 small Hamiltonians are independent of t. We only really need to show this for the cases when the index of critical points differ by 1 or 2 (all we need to define the chain complexes). In showing this, we will have shown that the Floer and Morse trajectories coincide for such Hamiltonians.

When the indices differ by 1, take $H_k := H/k$, $k \in \mathbb{Z}^+$. Suppose in each case, we have a trajectory u_k dependent on t. We can then produce a v which is the limit for a translated sequence of the u_k. Showing v is independent of t would give us a contradiction. v has the property of being periodic in t with period $1/k$ for all k. This forces v to be independent of t, giving us the proper contradiction.

In the case of indices differing by 2, we use some gluing techniques to show that the Floer gluing and Morse gluing coincide eventually by the uniqueness of gluing. This concludes the proof to show that Morse and Floer trajectories coincide in this autonomous C^2 small Hamiltonian case. Therefore, the main theorem 10.1.1. is proved.