An Overview of Seiberg-Witten Theory

Sam Auyeung

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1 $Spin^c$ Structures and the Dirac Operator

1.1 $Spin^c$ Structures

Let (V, q) be a real, finite dimensional, inner product space; $Spin(V)$ may be defined as the double cover of $SO(V)$ or we may instead define it using Clifford algebras. Let $Pin(V)$ be the group generated by elements $v \in V$ with $\|v\|^2 = 1$ and $Spin(V) = Pin(V) \cap Cl_0(V)$.

We may define $Spin^c(V)$ using the complexified Clifford algebra $Cl(V) \otimes \mathbb{C}$; it is the subgroup generated by $Spin(V)$ and $S^1 \subset \mathbb{C}$.

Example 1.1. $Spin(1) = \{\pm 1\}, Spin(2) = S^1, Spin(3) = S^3 = SU(2), Spin(4) = SU(2) \times SU(2)$. $Spin(n)$, for $n > 2$ is simply connected. The group that interests us is $Spin^c(4) = (SU(2) \times SU(2)) \times_{\mathbb{Z}_2} S^1$.

An oriented Riemannian manifold (X^n, g) gives us a natural $SO(n)$ bundle: we consider the tangent bundle TX and since the manifold is oriented, we can consider oriented orthonormal bases on each T_pX; $SO(n)$ acts on these bases and thus, we get an $SO(n)$ bundle, called the frame bundle F of TX.

A *Spin* or $Spin^c$ structure on X is not merely a principal bundle but one that **lifts** $F \to X$. There is a natural map $Spin(n) \to SO(V)$ which is really just quotienting by $\mathbb{Z}_2$. Thus, a spin structure $P \to X$ is a $Spin(n)$ bundle which, when we quotient by $\mathbb{Z}_2$, is isomorphic to F. Similarly, there is a natural map from $Spin^c(n) \to SO(n) \times S^1 \to SO(n)$ which arises first by modding out $\mathbb{Z}_2$ diagonally and then modding out S^1. So a $Spin^c$ structure $\widetilde{P} \to X$ is one which quotients by the above construction to an $SO(n)$ bundle isomorphic to $F \to X$. If X admits a spin or $Spin^c$ structure, we call it a spin or $Spin^c$ manifold.

Note that we are assuming that X is orientable; we don't discuss spin or $Spin^c$ outside of this assumption. There is a well known obstruction theory result for spin manifolds. X is spin if and only if $w_2 = 0$ (the 2nd Stiefel-Whitney class). By the way, if X is a closed spin 4-manifold, its intersection form is even. If X is simply connected, the converse holds.

We may also ask when X is $Spin^c$. From a $Spin^c$ structure $P \to X$, we can construct a complex line bundle which we call the determinant line bundle L. It can be constructed in the following way. It is well-known that principal G-bundles over a manifold X are classified as pullbacks of EG by some maps $X \to BG$. More precisely, the isomorphism classes of G-bundles over X are in 1-1 correspondence with homotopy classes of maps $X \to BG$. Now, there is a natural map $\det : Spin^c(n) \to S^1$ which comes from letting $Spin(n)$ be the kernel. In some sense, because of the $\mathbb{Z}_2$, this map is like squaring. It induces a map $\det : BSpin^c(n) \to BS^1$ which we call by the same name. Thus, if $P \cong f^* ESpin^c(n)$, then $L \cong (\det \circ f)^* ES^1$.

Now we can describe an obstruction theory result. If X admits a $Spin^c$ structure, then when we consider $c_1(L)$, it reduces mod 2 to $w_2(X)$. Conversely, if $L \to X$ is a complex line bundle with $c_1(L) = w_2(X) \pmod 2$, then there exists a $Spin^c$ structure P with $\det P \cong L$.

Claim: All closed, oriented Riemannian 4-manifolds admit a $Spin^c$ structure.

1.2 Spin Bundle

Now, we can, to any G-bundle, associate to it a vector bundle, once we fix a vector space V and a linear representation $\rho : G \to Aut(V)$. We consider the total space $P \times V$ and mod out in the following way: $(p \cdot g, v) \sim (p, g \cdot v)$.

There are, of course, many representations to choose from in the case of $Spin(V)$. However, it is natural to study finite dimension irreducible complex representations of $Cl(V)$ restricted to a complex representation of $Spin(V)$. It turns out that there is only one such restriction, up to isomorphism (of course, once we restrict, the representation might not remain irreducible). Call it $S_{\mathbb{C}}(V)$. We can also consider how $Cl(V) \otimes \mathbb{C}$ acts on $S_{\mathbb{C}}(V)$. Moreover, if $\dim V = 2n$, this $Spin(V)$ representation decomposes into irreducible complex representations $S^{\pm}_{\mathbb{C}}(V)$ of dimension 2^n by the action of $\omega_{\mathbb{C}}$ (recall $\omega_{\mathbb{C}}^2 = 1$). If $\dim V = 2n+1$, then the representation is already irreducible and of dimension 2^n.

Moreover, there is a unique way to extend these $Spin(V)$ representations to $Spin^c(V)$ representations; it respects the decomposition when V is even dimensional.

1.3 The Dirac Operator $\partial\!\!\!/_A$

Let us state once and for all our setting: we consider a fixed $Spin^c$ structures $P \to X$ where X is a closed, oriented Riemannian 4-manifold. Let $S_{\mathbb{C}}(P)$ be the vector bundle associated to P with vector space $\mathbb{R}^4$ and representation being the unique one that extends to $Cl(\mathbb{R}^4) \otimes \mathbb{C}$. It decomposes into two irreducible complex vector bundles, each of complex rank 2.

We wish to construct an operator on sections of $S_{\mathbb{C}}(P)$ using covariant derivatives and Clifford multiplication.

The covariant derivative must come from a connection. There's a natural connection to choose from on an $SO(4)$ bundle: the Levi-Cività connection. It's the unique connection which is orthogonal and also torsion-free. It easily lifts to being a connection on a $Spin(4)$ bundle because a connection can be viewed as a 1-form with values in the Lie algebra and conveniently, $\mathfrak{so}(4) \cong \mathfrak{spin}(4)$. On the line bundle $L = \det P$, we can choose a connection A. Thus, from these two, we can form a connection on a principal $SO(4) \times S^1$ which we can lift to our $Spin^c$ bundle P as $Spin^c(4) \to SO(4) \times S^1$ is a finite cover and thus they have the same Lie algebras. Let ∇ be the induced covariant derivative on $S_{\mathbb{C}}(P)$.

Now, let $\{e_1, ..., e_4\}$ be an oriented orthonormal frame for T_pX. We define the **Dirac operator** $\partial\!\!\!/_A : C^\infty(S_{\mathbb{C}}(P)) \to C^\infty(S_{\mathbb{C}}(P))$ on smooth sections of the spin bundle. These are called **spinors**. The definition is:

$$\partial\!\!\!/_A(\sigma)(p) = \sum_{i=1}^{4} e_i \cdot \nabla_{e_i}(\sigma)(p)$$

where $\cdot$ means Clifford multiplication. Note that $\partial\!\!\!/_A$ interchanges the plus and minus parts of $S_{\mathbb{C}}(P)$ because of the Clifford multiplication. As a reminder, if ∇ is a connection on a vector bundle $E \to X$, its image is in sections of $T^*X \otimes E$. However, if Y is a vector field of X, then ∇_Y maps sections of E to sections of E.

It can be shown that the Dirac operator is **elliptic** by considering its principal symbol which is Clifford multiplication by $i\xi$, $\xi \in T_x^*X$. Since X is closed, elliptic implies Fredholm and $\partial\!\!\!/_A$ is even self-adjoint. Therefore, Fredholm index of $\partial\!\!\!/_A$ on the sections of the full spin bundle is

0. However, when we restrict our attention to $\not{\partial}_A : C^\infty(S^+_{\mathbb{C}}(P)) \to C^\infty(S^-_{\mathbb{C}}(P))$, the index can be computed by the Atiyah-Singer Index theorem to be: $(c_1(L)^2 - 2\chi - 3\sigma)/4$. Here, χ is the Euler characteristic of X and σ is the signature of X.

2 The Seiberg-Witten Equations

The Seiberg-Witten Equations are a pair:

$$\not{\partial}_A \psi = 0; \qquad F_A^+ = q(\psi) := \psi \otimes \psi^* - \frac{|\psi|^2}{2} \operatorname{id}.$$

The first equation is pretty self explanatory; such ψ that satisfy the Dirac equation are called **harmonic spinors** and by elliptic regularity, even if ψ is of Sobolev type, say $W^{2,2}$, it will turn out to be smooth (actually we can say something stronger; it has an analytic continuation result). Also, by self-adjointness, harmonic spinors are orthogonal to the image of $\not{\partial}_A$.

The second equation needs a bit more explanation. F_A^+ is the self-dual part of the curvature 2-form of a $U(1)$ connection A (the self-dual comes from $\omega_{\mathbb{C}}$ which behaves a lot like Hodge-*). On the RHS, we have $\psi \otimes \psi^*$ which is an element of $C^\infty(S_{\mathbb{C}}(P) \otimes S_{\mathbb{C}}(P)^*) \cong C^\infty(\operatorname{End}(S_{\mathbb{C}}(P)))$. It's clear that the trace of $\psi \otimes \psi^*$ is $|\psi|^2$ and so subtracting off the second term makes it a traceless endomorphism.

How does this relate to curvature 2-forms? The endomorphisms on the spin bundle can be identified with 2-forms because there is a way to define an action of differential forms on the spin bundle. First, recall that the Clifford algebra, as a vector space, is the exterior algebra with a new type of multiplication. Thus, differential forms act on the spin bundle; and this particular $q(\psi)$ can be realized as a section of $\Lambda^2_+(T^*X) \otimes \mathbb{C}$; then identify TX and T^*X by the Riemannian metric g.

2.1 Configuration Space

In order to consider a solution (A, ψ) to the SW equations, we consider the space in which they live. We call this space the **configuration space** $\mathcal{C}(P) = \mathcal{A}_{2,2}(L) \times W^{2,2}(S^+(P))$; that is, we're taking them to be Sobolev elements in $W^{2,2}$ rather than smooth elements.

As is typical, we let $F : \mathcal{C}(P) \to W^{1,2}((\Lambda^2_+ T^*X \otimes i\mathbb{R}) \oplus S^-(P))$ be defined by $F(A, \psi) = (F_A^+ - q(\psi), \not{\partial}_A \psi)$. So $F^{-1}(0)$ are the solutions.

2.2 Change of Gauge

However, the space of solutions is enormous and we'll like to consider them up to some equivalence. The group of changes of gauge $\mathcal{G}(P)$ will be what we quotient our configuration space by. These are bundle automorphisms of P which cover the identity on the frame bundle of the tangent bundle. That means when we quotient $P \to X$ to obtain the frame bundle $F \to X$, our bundle automorphism is just the identity map. The automorphism only acts on the S^1 piece and possibly on the $Spin(4)$ part but in a way compatible with the map $Spin(4) \to SO(4)$.

It turns out that the $\mathcal{G}(P)$ is an infinite dimensional abelian Lie group. Its Lie algebra is the space of $W^{3,2}$ sections of the trivial bundle $X \times i\mathbb{R}$ with trivial bracket.

The action of $\mathcal{C}(P)$ is somewhat natural. Let $\sigma \in \mathcal{G}(P)$. Then σ induces bundle endomorphisms $\det \sigma$ and $S^\pm(\sigma)$ on L and $S^\pm(P)$. If we view $\det \sigma$ as a map from $X \to S^1$, then its simply the image of σ under squaring from $S^1 \to S^1$. The action is given by $(A, \psi) \cdot \sigma = ((\det \sigma)^* A, S^+(\sigma^{-1})(\psi))$. We also have an action on $W^{1,2}((\Lambda^2_+ T^*X \otimes i\mathbb{R}) \oplus S^-(P))$: σ acts trivially on the first factor and by $S^-(\sigma^{-1})$ on the second factor.

One can show that $F((A,\psi)\cdot\sigma) = F(A,\psi)\cdot\sigma$; i.e. F is $\mathcal{G}(P)$-equivariant. Thus, if (A,ψ) is a solution to the SW equations, so is $(A,\psi)\cdot\sigma$.

It is important to make a point here that the stabilizer of an element (A,ψ) is trivial unless $\psi = 0$, in which case the stabilizer can be identified naturally with S^1 (as groups). Thus, we'll see that when we quotient $\mathcal{C}(P)$ by $\mathcal{G}(P)$, we'll get some cones over $\mathbb{C}P^\infty$ where the vertex is something of the form $(A,0)$. This is because the link of $(A,0)$ is something like S^∞ and we have the following diagram.

$$S^1 \hookrightarrow S^\infty \longrightarrow \mathbb{C}P^\infty$$

3 The Seiberg Witten Moduli Space

The reason the Seiberg-Witten equations are so useful is that the moduli space $\mathcal{M}(P)$ for a *Spin*^{c} structure P is actually a **closed orientable smooth manifold** in generic cases. (X,g) will be our closed, oriented smooth Riemannian 4-manifold and P, our *Spin*^{c} structure. First, we have the following theorem:

Theorem 3.1. *Let (a_n,ψ_n) and (b_n,μ_n) be solutions of the SW equations and suppose they converge to (a,ψ) and (b,μ) in the $W^{2,2}$ topology. Also suppose there is a sequence $\sigma_n \in \mathcal{G}(P)$ such that $(a_n,\psi)\cdot\sigma_n = (b_n,\mu_n)$. Then there exists a subsequence of σ_n with limit σ such that $(a,\psi)\cdot\sigma = (b,\mu)$.*

The idea of the proof is as follows. Let $\tau_n = (\det\sigma_n)^*$. Then $\tau_n a_n = \tau_n^{-1} d\tau_n + \tau_n^{-1} a_n \tau_n = b_n$. Since the groups are abelian, we have $\tau_n^{-1} d\tau_n = b_n - a_n$ or $d\tau_n = \tau_n(b_n - a_n)$. We get natural bounds on a_n, b_n and a bound on τ_n will thus give a bound on $d\tau_n$. Now elliptically bootstrap. This result shows that the moduli space (so we quotient by change of gauge) is **Hausdorff** in C^∞.

3.1 Compactness of the Moduli Space

Compactness of the moduli space requires some analytic estimates. However, we get a stronger compactness: for any metric on X, there are only finitely many nonempty moduli spaces (arising from *Spin*^{c} structures) and each is compact. Of course, we're dealing with those that have nonnegative formal dimension. This result follows from two phenomena; we'll explain it more in Proposition 3.3.

1. Uhlenbeck showed that in general, L^2 bounds on curvature of the connections imply $W^{1,2}$ bounds on the connections in an appropriate gauge.

2. We have *a priori* pointwise bounds on the spinor, and hence, pointwise bounds on the curvature of any solution.

The Weitzenböck identity is the main tool in obtaining bounds on solutions to the Seiberg-Witten equations (need to play with Ricci tensors and indices to prove it). The identity is:

$$\partial\!\!\!/_A^2\psi = \nabla_A^*\nabla_A\psi + \frac{\kappa}{4}\psi + \frac{F_A}{2}\cdot\psi.$$

If ψ is a harmonic spinor, then the LHS is zero. Let's use the fact that ψ is a $+$-spinor so $F_A\cdot\psi = F_A^+\cdot\psi$. Then, since $F_A^+ = q(\psi)$, we get $|F_A|/2 = |\psi|^2/4$; this is an easy computation using local representations. Finally, we have an equation when we take the above Weitzenbock equation and take an inner product with ψ:

$$|\nabla_A \psi|^2 + \frac{\kappa}{4}|\psi|^2 + \frac{|\psi|^4}{4} = 0.$$

Then, we can show that for every $x \in X$, $|\psi(x)|^2 \leq \kappa_X^-$. Here, we have $\kappa^-(x) = \max\{-\kappa(x), 0\}$ and $\kappa_X^- = \max_{x\in X} \kappa^-(x)$. Thus, we have a uniform bound on the spinor fields and a nice additional lemma.

Lemma 3.2. *If X admits a metric with nonnegative scalar curvature, then the only solutions are reducible because $|\psi|^2 \leq 0$.*

In particular, T^4 is flat and so admits a metric with zero scalar curvature. Thus, we'll see this means $SW_{T^4} \equiv 0$.

More over, $c_1^2(L) - 2\chi - 3\sigma \geq 0$ by our assumption of non-negative formal dimension. But also, $c_1^2(L) = \frac{1}{4\pi^2}(|F_A^+|^2 - |F_A^-|^2)$; the $-$ sign comes from the fact that F_A^- is anti-self dual. So we obtain L^2 bounds on the curvature $F_A = F_A^+ + F_A^-$. The bounds only depend on scalar curvature κ, the Euler characteristic, and signature of X. Note that this gives us the following result:

Proposition 3.3. *For a given metric g on X, there are only finitely many $Spin^c$ structures which admit a non-empty moduli space with non-negative formal dimension.*

The proof goes as follows. A solution (A, ψ) to the SW equations gives us a bound on F_A. Thus, $c_1 = [iF_A/2\pi] \in H^2(X, \mathbb{R})$ is in a bounded set. Moreover, it is an integral class so it is discrete. Thus, there are finitely many possibilities for the determinant line bundle of a $Spin^c$ structure if we're looking for non-empty moduli spaces with non-negative formal dimension.

Now, combining the L^2 bounds on curvature with a result of Uhlenbeck gives us $W^{1,2}$ bounds on the connections in an appropriate gauge. We prove a local slice theorem to show that in fact, we can always ensure that our connections are in some appropriate gauge to get the $W^{1,2}$ bound. Through elliptic bootstrapping, we show our solutions are C^∞. Thus, we have C^∞ bounds on the connections. This, together with the pointwise bounds on the harmonic spinors gives us compactness. Therefore, if we have a sequence of solutions, we can find a subsequence that converges to another solution. Thus, the moduli space is compact.

Local Slice Theorem: In a brief word, the local slice theorem is something of a tubular neighborhood theorem. If we have Lie group G acting on M, then for some $x \in M$, $S := G \cdot x$, the orbit, is hopefully a submanifold. Moreover, we'll like to say it's diffeomorphic to G/G_x (G mod the stabilizer of x). What we get is that if we realize G/G_x as the zero section of $G \times_{G_x} (TX/TS)$, then there is a neighborhood U of G/G_x which is diffeomorphic to a neighborhood V of S and is an equivariant extension of the map $[g] \mapsto g \cdot x$.

3.2 Smoothness of the Moduli Space

We will show that away from reducible solutions to the SW equations, the moduli space defined is a smooth manifold because we know precisely that the singular points arise from reducible solutions.

However, ultimately, we want to ensure there are no reducible solutions. Thus, we will first introduce the perturbed SW equations. Then, we show that away from reducible solutions for these new equations, the moduli space is a smooth manifold. Lastly, we will show that we can pick a generic perturbation so that the moduli space does not contain any reducible solutions and thus is smooth everywhere.

Step 1: The perturbed SW equations only changes the curvature equation. We request $F_A^+ = q(\psi) + i\eta$ for an $\eta \in \Lambda_+^2(X, \mathbb{R})$. So we're perturbing by a self-dual, purely imaginary 2-form $i\eta$. Let

$$F : W^{3,2}(\Lambda_+^2(X, \mathbb{R})) \times \mathcal{C}_4(P) \to W^{3,2}(\Lambda_+^2(X, i\mathbb{R}) \oplus S^-(P)),$$
$$F(A, \psi, h) = (F_A^+ - q(\psi) - ih, \not{\partial}_A \psi)$$

Observe that in the h coordinate, dF is basically inclusion and also then, surjective.

Step 2: To show that the moduli space is smooth, we would like to use an implicit function theorem. If we have a regular value of our SW map, then its preimage forms a smooth manifold.

To do this, we should show that F and the gauge action are Fredholm. This is not too hard because the equations are elliptic and linear elliptic maps on a closed manifold X are Fredholm.

Next, we show that at any point where $F(\eta, A, \psi) = 0$ and $\psi \neq 0$, when we project to vertical fibers, $G := dF \circ \pi$ is surjective. This is the **transversality** property. From our observation above about dF being an inclusion on the first factor, we really just need to show G is surjective on the second factor. So let's restrict dF to the second factor and then project; continue to call this map G. We now have a formula: $G(a, \eta) = \not{\partial}_A \eta + \frac{1}{2} a \cdot \psi$.

To prove surjectivity, suppose we have an element $Z \in S^-(\widetilde{P})$ orthogonal to the image of G; we suppose $Z \neq 0$. Being orthogonal to the image of dF means Z is orthogonal to the image of $\not{\partial}_A$. But $\not{\partial}_A$ is self-adjoint so $\not{\partial}_A Z = 0$. That is, Z is a S^--harmonic spinner and so is C^∞ by elliptic regularity. But not only this, we have that Z has an analytic continuation property and thus, is non-vanishing in an open set. Let U be a neighborhood where Z and ψ don't vanish and let $x_0 \in U$.

Claim: We can produce an element Y in the image of G and show that $\langle Y, Z \rangle \neq 0$; a contradiction. This would show that G is surjective at solutions to the perturbed SW equations. So let's show there is such a Y.

Recall that $\mathbb{R}^4 \otimes \mathbb{C} \cong Cl_1(\mathbb{R}^4)^+ \cong \mathrm{Hom}(S^+, S^-)$. Let's suppose we're given $\sigma^\pm \in S^\pm$. We can define a morphism $\varphi : S^+ \to S^-$ that takes σ^+ and maps it to something that is not orthogonal to σ^-; i.e. $\langle \varphi(\sigma^+), \sigma^- \rangle \neq 0$. Let ψ, Z be our $\sigma^\pm$. By the identification, φ corresponds to some element $a \in \mathbb{R}^4 \otimes$; we can treat $a \in \mathbb{R}^4 \cong T^*_{x_0} X$ if we just want $\mathrm{Re}\langle a \cdot \psi(x_0), Z(x_0) \rangle > 0$. Now extend a to be a 1-form on X with compact support in U. Thus, $\int_X \langle a \cdot \psi, Z \rangle \neq 0$ which is the contradiction.

We conclude from this that (η, A, ψ) is a regular value and the moduli space $\mathcal{M}^*(P, \eta) = F^{-1}(0)$ is a smooth manifold by the implicit function theorem.

Step 3: Let $\mathcal{M}^*(P)$ be the space of triples (η, A, ψ) such that $[A, \psi]$ are solutions to the SW equations perturbed by η. Let $\pi : \mathcal{M}^*(P) \to W^{3,2}(\Lambda_+^2(X, i\mathbb{R}))$ be projection; it is a smooth map. Clearly, $\pi^{-1}(\eta) = \mathcal{M}^*(P, \eta)$. Now, the fact that F is Fredholm shows that π is also Fredholm. Thus, we may use Sard-Smale (the theorem requires Fredholm) to show that the regular values of π are dense. Thus, we've shown that these perturbations which give a smooth manifold (without boundary) of irreducible solutions are indeed **generic**.

3.3 Reducible Solutions

In the previous section, we see that away from reducible solutions, the moduli space is a smooth manifold. What about the reducible solutions? First, in a neighborhood of a reducible solution in the configuration space, the boundary is something like S^∞ and its stabilizer is S^1. Thus, when we quotient, the reducible solutions give us singular points which are cones over $\mathbb{C}P^\infty$.

Notice that if there are reducible solutions $(A, 0)$ then in the standard equations, they satisfy $F_A^+ = 0$; i.e. A is anti self-dual. We'll show that by perturbing the SW equations, there are no reducible solutions. We will, in later sections, prove that under small perturbations, the SW invariant is in fact invariant.

Now, in general, $F_A = -2\pi i c_1(L)$. This means the curvature is independent of A. In the perturbed setting, a reducible solution $(A, 0)$ satisfies $F_A^+ = i\eta$. Because of the relationship of curvature with c_1, we only have reducible solutions when, after orthogonal projection to the self-dual harmonic 2-forms, $\eta^+ = 2\pi c_1(L)^+$. Our hope is that by perturbing, our η fails this equality.

Question: Can we always perturb so that the equality here fails? If so, is the perturbation generic or are there relatively few ways to perturb?

Answer: Yes, we can always perturb and the perturbations making the equality fail are generic. However, we'll be needing the assumption that $b_2^+ > 0$.

Theorem 3.4. *Suppose that $b_2^+(X) > 0$ and that for the $Spin^c$ structure P, its determinant line bundle L is such that $c_1(L)$ is not a torsion cohomology class. For a generic metric, there are no reducible solutions to the Seiberg-Witten equations coming from P. Moreover, for any $Spin^c$ structure, if there are no reducible solutions to the SW equations, then for all sufficiently small perturbations of the equations, there are no reducible solutions either.*

The proof relies on Taubes' result: if there exists a reducible solution, the Riemannian metric g must be of a particular type, living in a closed codim b_+^2 subspace in the space of all Riemannian metrics on X. So when $b_+^2 > 0$, the set of metrics which **do not** admit reducible solutions is open and dense. Moreover, there are only finitely many $Spin^c$ structures with nonempty moduli spaces of non-negative formal dimension. Therefore, we can take a finite intersection of all these generic metrics for each $Spin^c$ structure P and still have a dense set of generic metrics for which there are no reducible solutions coming from any $Spin^c$ structure.

We may say even more; for a generic metric, if the SW equations have no reducible solutions for any $Spin^c$ structure, then any sufficiently small perturbation will also lack reducible solutions.

Theorem 3.5. *For any metric and a generic perturbation, there are no reducible solutions.*

The proof is as follows: for any metric g, a reducible solution to the perturbed equations satisfies $F_A^+ = \eta$. Moreover, it must be that the orthogonal projection of η into the self-dual harmonic two-forms equals the projection of $2\pi c_1$ into the same space (note that the orthogonal projection depends on our Riemannian metric g). If we perturb η slightly, this equality fails.

Corollary 3.6. *For any metric and a generic perturbation, there are no reducible solutions and hence the moduli space of all solutions is a smooth manifold.*

3.4 Orientation of the Moduli Space

In order to define the invariant, we need to integrate the moduli spaces; thus, we need them to be orientable. Consider the theorem:

Theorem 3.7. *The open subset of smooth, irreducible points of $\mathcal{M}(P)$ is an orientable manifold. A choice of orientations of $H^0(X, i\mathbb{R})$, $H^1(X, i\mathbb{R})$, and $H_+^2(X, i\mathbb{R})$ determine an orientation of $\mathcal{M}(P)$ at any irreducible point.*

This theorem establishes that the moduli spaces are orientable and that there is a canonical way to do so. We'll take the proof in steps but we first need a definition.

Definition 3.8. *The determinant bundle of a linear Fredholm operator $F : V \to W$ is defined as* $\det F = \Lambda^{top} \ker F \otimes (\Lambda^{top} coker\ F)^*$*. There's a way to glue these together for a family of Fredholm operators F_s over some parameter space S and obtain a determinant line bundle for the family as a line bundle over S.*

Our parameter space will be $\mathcal{C}(P)$. There is a lemma that will be useful:

Lemma 3.9 (6.6.1)**.** *Suppose F_s and F'_s are two families of Fredholm operators parametrized by S. Further, suppose they are homotopic. Then the homotopy determines an isomorphism between the determinant line bundles $L \to S$ and $L' \to S$.*

Proof of Orientability

1. Let $(A, \psi) \in \mathcal{C}(P)$ and consider the elliptic complex which comes from linearizing the Seiberg-Witten map F:

$$0 \longrightarrow W^{3,2}(X; i\mathbb{R}) \xrightarrow{D_1} W^{2,2}((T^*X \otimes i\mathbb{R}) \oplus S^+(P))$$

$$\xrightarrow{D_2} W^{1,2}((\Lambda^2_+ T^*X \otimes i\mathbb{R}) \oplus S^-(P)) \longrightarrow 0$$

   Here, $D_1 = (2d, - \cdot \psi)$ and

$$D_2 = \begin{pmatrix} d^+ & -Dq_\psi \\ \cdot \frac{1}{2}\psi & \not{\partial}_A \end{pmatrix}.$$

2. When studying elliptic complexes, we study the symbol of operators which look at the highest order terms. Since $\not{\partial}_A$ is a 1st order linear operator, we can study just the 1st order terms and do away with the 0th order terms. That is, we homotope the operator $D = (D_2, D_1^*)$ to an operator $E = (E_2, E_1^*)$ which does not have any 0th order terms. What we have is

$$E_1^* = (2d^*, 0), \quad E_2 = \begin{pmatrix} d^+ & 0 \\ 0 & \not{\partial}_A \end{pmatrix}.$$

3. By lemma 6.6.1, $\det D \cong \det E$. Both are bundles over $\mathcal{C}(P) \times I$.

4. $\mathcal{G}(P)$ acts on these bundles "nicely" so after restricting to $\mathcal{C}^*(P) \times I$ and quotienting, we get line bundles over $\mathcal{B}^*(P) \times I$. Denote the one coming from E by $\xi \to \mathcal{B}^*(P) \times I$.

5. Claim: ξ is trivial and has orientation determined by the line bundle

$$\Lambda^{top} H^1 \otimes (\Lambda^{top} H^2)^{-1} \otimes (\Lambda^{top} H^0)^{-1}.$$

   Proof. We can restrict our attention to just showing the fiber over $\mathcal{B}^*(P) \times \{1\}$ is trivial. Now, over $\mathcal{C}^*(P) \times \{1\}$, the family of Fredholm operators split as $d^+ + \not{\partial}_A + 2d^*$; the only family changing as we parametrize over $\mathcal{C}^*(P)$ is $\not{\partial}_A$, a family of complex operator. Thus, $\xi_1 = \mathbb{R} \otimes \det(d^+ + 2d^*) \otimes \det_{\mathbb{R}} \not{\partial}_A$. Quotienting by $\mathcal{G}(P)$ preserves the tensor structure and also the complex structure of $\det_{\mathbb{R}} \not{\partial}_A$. Moreover, the real determinant line bundle of any family of complex operators is trivial and oriented by the complex structure.

   So we only need to orient $\det(d^+ + 2d^*)$ over a point. Let $\mathcal{H}$ denote harmonic forms. Observe that $\ker d^+ = \mathcal{H}^1 \oplus \operatorname{Im} d \oplus \{\text{maybe something else}\}$. On the other hand, $\ker d^* =$

$\mathcal{H}^1 \oplus \operatorname{Im} d^*$. Nothing in the image of d is in this kernel. For if, say $df \in \ker d^*$ where $f : X \to \mathbb{R}$, then $0 = \langle d^* df, f\rangle = \langle df, df\rangle = |df|^2 = 0$. Thus, $df = 0$. This shows that $\ker(d^+ + 2d^*) = \mathcal{H}^1 \cong H^1(X, i\mathbb{R})$, the intersection of the two kernels. Also, $\Lambda^0 = \mathcal{H}^0 \oplus \operatorname{Im} d^*$ so $\operatorname{coker} d^* = \mathcal{H}^0 \cong H^0(X; i\mathbb{R})$ while $\operatorname{Im} d^+$ is the space of exact self-dual 2-forms. So $\operatorname{coker} d^+ = \Lambda^2_+/\operatorname{Im} d^+ = H^2_+(X; i\mathbb{R})$. So $\operatorname{coker}(d^+ + 2d^*) = H^0(X; i\mathbb{R}) \oplus H^2_+(X; i\mathbb{R})$ and the claim is proved. $\square$

6. Orienting $\xi \to \mathcal{B}^*(P) \times I$ amounts to orienting $\det D \to \mathcal{B}^*(P) \times I$ when (A, ψ) is an irreducible solution. Now $\ker D_{(A,\psi)} = T_{(A,\psi)}\mathcal{M}(P)$ and $D_{(A,\psi)}$ is generically surjective. Thus, orienting $\det D_{(A,\psi)}$ orients $T_{(A,\psi)}\mathcal{M}(P)$. This orientation comes canonically from H^0, H^1, H^2.

3.5 Variations of the Moduli Space

The main goal of this section is to study what happens to our moduli spaces when we choose different metrics and perturbations. We will use the fact that the space of metrics on a manifold is convex and thus, we can find a path through the space of generic metrics and regular perturbations connecting any two pairs $(g_0, \eta_0), (g_1, \eta_1)$ which lifts to a cobordism between the moduli space at (g_0, η_0) and the moduli space at (g_1, η_1).

Let $\mathcal{M}(P, \eta)$ be elements $([A, \psi], t) \subset \mathcal{B}(P) \times [0, 1]$ satisfying

$$F_A^{+_t} = q(\psi) + i\eta(t); \qquad \not{\partial}_{A,g_t}(\psi) = 0.$$

Here, $\not{\partial}_{A,g_t}$ is the Dirac operator defined by the Levi-Civita connection for g_t and the connection A on L. Similarly, $F_A^{+_t}$ is the self-dual part of the curvature form of A; of course, the Hodge-* operator now depends on how g_t varies.

Let $\mathcal{P}(\eta_1, \eta_2)$ be the space of $W^{1,2}$ paths $\eta : [0, 1] \to W^{3,2}(\Lambda^2_+ T^*X \otimes \mathbb{R})$ satisfying $\eta(0) = \eta_0$ and $\eta(1) = \eta_1$. Since we can integrate along paths, $\mathcal{P}(\eta_1, \eta_2)$ is a Hilbert manifold with tangent space at $\eta(t)$ being the space of $W^{1,2}$ paths $\eta : [0, 1] \to W^{3,2}(\Lambda^2_+ T^*X \otimes \mathbb{R})$ which vanish at the endpoints. Define

$$\mathbf{F} : \mathcal{C}^*_4(P) \times I \times \mathcal{P} \to W^{3,2}(\Lambda^2_+ T^*X \otimes i\mathbb{R} \oplus S^-(P))$$

by $\mathbf{F}(A, \psi, t, \eta) = (F_A^{+_t} - q(\psi) - i\eta(t), \not{\partial}_{A,g_t}(\psi))$. The claim is that we have a transversality result here as well: $d\mathbf{F}$ is surjective at every (A, ψ, t, η) on which $\mathbf{F}$ vanishes. Let $\mathbf{M}^*(P)$ be the moduli space of irreducible solutions parametrized by $\mathcal{P}$; i.e. $\mathbf{F}^{-1}(0)$ quotiented by $\mathcal{G}_5(P)$. This is a smooth manifold with boundary.

The projection $\mathbf{M}^*(P) \to \mathcal{P}$ is a smooth map. Sard-Smale gives us that regular values $\eta \in \mathcal{P}$ are generic and thus, the fiber over η is a smooth manifold with boundary. It is precisely $\mathcal{M}(P, \eta)$. This is compact as each $\mathcal{M}(P, \eta(t))$ is compact. Thus, what we've showed is that for a generic path η as described above, $\mathcal{M}(P, \eta)$ is a compact, properly embedded, smooth submanifold of $\mathcal{B}^*(P) \times I$. Its boundary is the disjoint union $\mathcal{M}(P, \eta_0) \coprod -\mathcal{M}(P, \eta_1)$ (oriented).

In the above, we were not requiring smooth paths but rather $W^{1,2}$ paths. However, there is a dense set of C^∞ paths such that we still get this smooth cobordism.

4 The Seiberg-Witten Invariant

4.1 $b_2^+(X) > 1$

Let us assume that $b_2^+(X) > 1$. The Seiberg-Witten invariant of a 4-manifold satisfying this condition is given by the homology class of the moduli space $\mathcal{M}(P, \eta)$ of solutions in the configuration space mod gauge changes; it has dimension d. This configuration space $\mathcal{B}^*(P)$ is homotopy equivalent to $\mathbb{C}P^\infty \times K(H^1(X,\mathbb{Z}),1)$, a classifying space of $(S^1)^X := \{f : X \to S^1\}$ bundles. If $\pi_1 X = 0$, then its homotopy equivalent to $\mathbb{C}P^\infty$. The $H^*(\mathcal{B}(P);\mathbb{Z})$ has a canonical generator μ in even degrees. d is even if and only if $b_1 + b_2^+$ is odd (this also shows that the dimension of all the moduli spaces are either all even or all odd). In this case, by evaluating μ against the fundamental class of the Seiberg-Witten moduli space we obtain an integer

$$SW_{X,g,\eta}(P) = \int_{\mathcal{M}(P,\eta)} \mu^{d/2} \in \mathbb{Z}.$$

If d is odd, then we let $SW \equiv 0$ for all $Spin^c$ structures; this means that SW theory cannot tell us anything in the $b_1 + b_2^+$ is even case.

Even when d is even, *a priori*, the integer $SW(P)$ depends on the metric and perturbation. Let (g_1,η_1) and (g_0,η_0) be regular pairs of metrics and perturbations. By regular, we mean they yield smooth moduli spaces $\mathcal{M}(P,g_1,\eta_1)$ and $\mathcal{M}(P,g_0,\eta_0)$ (they do not contain any reducible solutions). From our previous discussion, we can form a smooth cobordism between these two spaces. Now, μ is closed. Thus, by Stokes' Theorem applied to the cobordism,

$$0 = \int_{\mathcal{M}(P,\eta)} d(\mu^{d/2}) = \int_{\mathcal{M}(P,g_1,\eta_1)} \mu^{d/2} - \int_{\mathcal{M}(P,g_0,\eta_0)} \mu^{d/2}.$$

Therefore, when $b_2^+ > 1$, the Seiberg-Witten invariant really is an **invariant** as it does **not** depend on a regular pair (g,η). The invariant is an **orientation-preserving diffeomorphism** invariant of the 4-manifold, and it has been used very effectively to distinguish many homeomorphic but not diffeomorphic 4-manifolds (exotic pairs).

4.2 $b_2^+(X) = 1$

When $b_2^+ = 1$, there is a codimension 1 space of bad metrics which forms a wall between two chambers in the space of all metrics. The wall is defined in the following way. First, let $\mathcal{R}$ be the space of all Riemannian metrics. The space of self-dual harmonic 2-forms, denoted $\mathcal{H}_+^2(X,\mathbb{R})$ has dimension $b_2^+ = 1$ and is thus, a line. Once we orient this line, to each metric g, we assign the unique g self-dual harmonic 2-form of norm 1, lying in the positive component of $\mathcal{H}_+^2(X,\mathbb{R})$ (with respect to orientation). Denote it by $\omega^+(g)$. Let $\omega^+(g)\cdot c_1(L)$ denote the intersection pairing of the two. The space of metrics g such that $\omega^+(g)\cdot c_1(L) = 0$ disconnects $\mathcal{R}$. Our two chambers are $\mathcal{R}_+$ which contains metrics g satisfying $\omega^+(g)\cdot c_1(L) > 0$ and $\mathcal{R}_-$ which is defined similarly.

Despite this wall, within each chamber $SW_{M,g,\eta}(P)$ stays constant for each $Spin^c$ structure P. So, within a chosen chamber, the invariant is indeed, an invariant.

4.3 Wall-Crossing Formula when $b_2^+(X) = 1$ and $b_1 = 0$

If in addition to $b_2^+ = 1$ we also know that $b_1 = 0$; e.g. $\pi_1 X = 0$, there is a wall-crossing formula describing the difference of SW in the two different chambers: $SW_-(P) = SW_+(P) + (-1)^{d/2}$.
Proof of the Wall-Crossing Formula:

1. Let $g_0 \in \mathcal{R}_-$, $g_1 \in \mathcal{R}_+$; we can make it so that this path crosses the wall only once at $g_{1/2}$. Similarly, we have perturbations $\eta(t)$ to make it so that $\mathcal{M}(P,t) := \mathcal{M}(P,g(t),\eta(t))$ is a smooth manifold for all $t \neq 1/2$.

2. Now, since $b_1 = 0$, the homotopy type of $\mathcal{B}^*(P) \simeq \mathbb{C}P^\infty \times K(H^1(X;\mathbb{Z}),1) = \mathbb{C}P^\infty$. Let $\mathbf{M} = \{([A,\psi],t) : [A,\psi] \in \mathcal{M}(P,t)\} \subset \mathcal{B}(P) \times I$ be the parametrized moduli space.

3. Claim: $\mathbf{M}$ has only one reducible solution at $t = 1/2$. That is, $\mathbf{M}$ is a singular cobordism between $\mathcal{M}(P, 0)$ and $\mathcal{M}(P, 1)$ and has only one singularity. I think I partially understand two proofs; of course, together, they do not equal fully understanding one proof. I'll record both:

   "Proof" 1 (doesn't seem to use $b_1 = 0$)

   Proof. We may choose our path in such a way that it crosses the wall transversely at time $t = 1/2$. Thus, the only possible reducible solutions live in $\mathcal{M}(P, 1/2)$. How do we guarantee there is only one? Well, for there to be a reducible, we need $F_A^+ = i\eta^+$ but $F_A^+ = -2\pi i[c_1^+]$. So we need $\eta^+ = -2\pi[c_1^+]$. This all lives inside of H_+^2 which is 1 dimensional by our $b_2^+ = 1$ assumption. Therefore, as we go along a path $\eta(t)$, we're going to hit $-2\pi[c_1^+]$ exactly once for each time we cross the wall. Since we're making sure the path is transverse to the wall, we get one reducible solution.

   Indeed, this transverse wall-crossing condition tells us that the singularity we get in $\mathcal{M}(P, 1/2)$ is **regular** in $\mathbf{M}$; in the t direction, we have the derivative of the parametrized Seiberg-Witten map being nonzero. It's sort of like an elliptic fibration $X \to \mathbb{P}^1$ with fibers being elliptic curves E. The fibers are generically nonsingular though you can have some singular fibers. Say the local coordinates on a singular fiber E is z and the local coordinates around this fiber in X are (z, w). So E is defined by some $F(z, w)$ locally. Then $\partial F/\partial z = 0$ at the singularity in E by definition but there's no reason *a priori* that $\partial F/\partial w = 0$. Hence, it can be regular in the full space. $\square$

   "Proof" 2

   Proof. We have the following fibration which gives rise to a long exact sequence of homotopy groups:

$$\begin{array}{ccc} \mathcal{G}(P) & \hookrightarrow & \mathcal{C}^*(P) \\ & & \downarrow \\ & & \mathcal{B}^*(P) \simeq \mathbb{C}P^\infty \end{array}$$

$$\pi_1\mathbb{C}P^\infty = 0 \longrightarrow \pi_0\mathcal{G}(P) \longrightarrow \pi_0\mathcal{C}^*(P) \longrightarrow \pi_0\mathbb{C}P^\infty = 0$$

   Now, $\mathcal{C}(P) = \mathcal{A}(\det P) \times \Gamma(S^+(P))$ and $\mathcal{A}$ is modeled on an affine space so is contractible while $\Gamma(S^+(P))$ is connected as X is connected. Hence $\mathcal{G}(P)$ is connected. This means that each element $\gamma \in \mathcal{G}(P)$ can be represented by some $f : X \to S^1$.

   Now, we want to see that there is a unique reducible connection. This part is rather hand-wavy and I'm not sure if it's correct, but morally, it seems it should be. If A satisfies $F_A^+ = -i\eta^+(1/2)$, then we have a reducible solution $(A, 0)$. Suppose that A' also satisfies this equation; so $F_A^+ = F_{A'}^+$. Expressing this in terms of their local connection 1-forms, we have $d\omega = d\omega'$ (the $\omega \wedge \omega$ part vanishes because $i\mathbb{R}$ is abelian). This means $\omega' = \omega + \alpha$ where α is a closed 1-form in $\Omega^1(L, i\mathbb{R})$. Here L is viewed as a $U(1)$-bundle rather than a complex line bundle.

   α is locally exact so we can represent it by some function. However, this function should somehow correspond to a function $f : X \to S^1$. In this way, we see that A and A' are gauge equivalent. Thus, up to gauge equivalence, there is only one solution to $F_A^+ = -i\eta^+(1/2)$ and $\not{\partial}_A\psi = 0$; namely this $(A, 0)$.

Note: Nicolescu claims that the reducible solution, though it is a singularity for $\mathcal{M}(P, 1/2)$, it is regular for $\mathbf{M}$. Not sure why... $\square$

4. This single reducible solution gives a cone over $\mathbb{C}P^{d/2}$ where d is the dimension of all the moduli spaces $\mathcal{M}(P,t)$, $t \neq 1/2$. Thus, cutting out the singularity, we get a cobordism with a "hole" in it. It has boundary $\mathcal{M}(P,1) - \mathcal{M}(P,0) + \mathbb{C}P^{d/2}$.

5. Somehow, we can show that μ is the negative of the generator of the cohomology ring $H^*(\mathcal{B}^*(P);\mathbb{Z})$ and so integrating $\mu^{d/2}$ over $\mathbb{C}P^{d/2}$ gives $(-1)^{d/2}$.

6. Hence,
$$0 = \int_{\mathbf{M}} d(\mu^{d/2}) = \int_{\mathcal{M}(P,1)} \mu^{d/2} - \int_{\mathcal{M}(P,0)} \mu^{d/2} + \int_{\mathbb{C}P^{d/2}} \mu^{d/2} = SW_+(P) - SW_-(P) + (-1)^{d/2}.$$

7. We conclude that the wall-crossing formula is: $SW_-(P) = SW_+(P) + (-1)^{d/2}$

What does this theorem say about the geometry? Suppose that $d = 0$ (which means that $(-1)^d = +1$) so we're literally just counting finitely many solutions. When you cross the wall, it says that generically, we lose exactly one solution. Hence, we have this -1 appearing. If $d > 0$, then we have, instead of a single solution vanishing, a whole $\mathbb{C}P^d$'s worth of solutions vanishing and by looking at orientation and all that, we have ± 1 appearing in the formula. Aleksander Doan says that if you want to understand this vanishing, you should study the 2nd cohomology of the elliptic complex and see that it vanishes. Somehow, studying the elliptic complex tells us something something obstructions, something something Kuranishi model.

So, by keeping track of a little more information about the chambers, it is still possible to use information from the Seiberg-Witten invariants to distinguish exotic pairs. For example, $\mathbb{C}P^2$ has $b_2^+ = 1$. We can look for exotic $\mathbb{C}P^2 \# n\overline{\mathbb{C}P}^2$ using a blow-up formula. In particular, when $n = 9$, we obtain infinitely many non-diffeomorphic smooth structures.

However, if $b_2^+ = 0$, then we can't say anything as then its not clear that the invariant is independent of the metric (and seems like it will be dependent). Thus, SW has nothing to say about S^4 and the smooth Poincaré conjecture for S^4 remains open.

4.4 An Involution in the Theory

It should be noted that we have complex conjugation on $Spin^c$ which preserves the $Spin$ piece but acts on S^1. Thus, given a $Spin^c$ bundle P and its determinant line bundle L, we may produce, using this complex conjugation, another $Spin^c$ bundle, call it $-P$. This is done by pulling back $P \to F \times L^*$ a map ι induced by complex conjugation.

$$\begin{tikzcd} & P \arrow[d] \\ F \times_X L \arrow[r, "\iota"] & F \times_X L^* \end{tikzcd}$$

Here, F is the frame bundle and L^* the dual of L. Using all the orientations from before, we have that $SW(-P) = (-1)^{\epsilon(X)} SW(P)$. $\epsilon(X) = (1 + b_2^+ - b_1)/2$ which is an integer because we're assuming $b_2^+ + b_1$ is odd and thus, $b_2^+ + b_1 - 2b_1$ is also odd.

5 Some Properties of the Seiberg-Witten Invariant

Here are some properties of the Seiberg-Witten invariant which is an integral function $SW : \mathcal{S}^c(X) \to \mathbb{Z}$ from the set of $Spin^c$ structures of a Riemannian 4-manifold. This section is taken from a survey paper of M. Hutchings and C. Taubes.

1. **(Invariance)** If $b_2^+(X) > 1$, then $SW(\widetilde{P})$ depends **only** on $\widetilde{P}$ and gives an **orientation-preserving diffeomorphism** invariant $SW : \mathcal{S}^c_X \to \mathbb{Z}$.

2. **(Naturality)** If X and Y are compact, oriented smooth 4-manifolds with $b^+ \geq 2$, $f : X \to Y$ is an orientation-preserving diffeomorphism, and $\widetilde{P} \in \mathcal{S}^c(Y)$, then $SW_X(f^*\widetilde{P}) = SW_Y(\widetilde{P})$.

3. **(Dimension)** Every **basic class** c (the 1st Chern class of the determinant line bundle of a $Spin^c$ structure that SW doesn't vanish on) of X satisfies $c \cdot c \geq 2\chi(X) + 3\sigma(X)$.

4. **(Finiteness)** $SW(\widetilde{P}) = 0$ for all but finitely many $\widetilde{P}$.

5. **(Symmetry)** There is a **charge conjugation** involution induced by conjugation $c \mapsto \bar{c}$ on $Cl(V) \otimes \mathbb{C}$. This involution acts on the set of $Spin^c$ structures (which sends $c_1(L)$ to $-c_1(L)$), and $SW(-\widetilde{P}) = (-1)^\epsilon SW(\widetilde{P})$ (ϵ is defined above in the involution subsection).

6. **(Blowup)** The Seiberg-Witten invariant for $Y = X\#\overline{\mathbb{C}P}^2$ contains similar information as that of X. We can sort of think that the original $Spin^c$ structures stick around after blow up but this needs to be made precise (see the statement below). $X\#\overline{\mathbb{C}P}^2$ will generally have additional $Spin^c$ structures coming from $\overline{\mathbb{C}P}^2$; let E be the (-1)-curve $\overline{\mathbb{C}P}^1 \subset \overline{\mathbb{C}P}^2$. This E contributes more $Spin^c$ structures. Thus, the full SW invariants differ.

   The more general statement is as follows:

   If X and Y have $b_2^+(X) \geq 2$ and $b_1(Y) = b_2^+(Y) = 0$ and $\widetilde{P}_Y \in \mathcal{S}^c(Y)$ and $\widetilde{P}_X \in \mathcal{S}^c(X)$ are $Spin^c$ structures whose characteristic classes $c = c_1(\det \widetilde{P}_X) \in H^2(X, \mathbb{Z})$ and $e = c_1(\det \widetilde{P}_Y) \in H^2(Y, \mathbb{Z})$ satisfy

   $$c \cdot c - 2\chi(X) - 3\sigma(X) + e \cdot e + b_2(Y) \geq 0$$

   then $SW_{X\#Y}(\widetilde{P}_X \# \widetilde{P}_Y) = SW_X(\widetilde{P}_X)$. In particular, the basic classes of $X\#Y$ have the form $c' = c + e$ where $c \in H^2(X, \mathbb{Z})$ is a basic class of X and $e \in H^2(Y, \mathbb{Z})$ is a characteristic vector (generator).

7. **(Connected Sum)** If $X = Y\#Z$ and $b_2^+(Y), b_2^+(Z) > 0$ then $SW_X \equiv 0$. (This does not contradict (6) because $b_2^+(\overline{\mathbb{C}P}^2) = 0$.)

8. **(Scalar Curvature)** If X has a metric of positive scalar curvature, then all the Seiberg-Witten invariants of X are zero.

9. **(Genus)** Let X have $b_2^+ \geq 2$ and $\Sigma \subset X$ be a compact connected oriented embedded 2-manifold which represents a nontorsion homology class. Suppose that $\Sigma \cdot \Sigma \geq 0$. Then the genus of Σ satisfies $2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |c \cdot \Sigma|$ for every basic class c of X.

10. **(Symplectic)** Let (X, ω) be a compact symplectic 4-manifold with $b_2^+ \geq 1$ and $\widetilde{P}_J \in \mathcal{S}^c(X)$ be the canonical $Spin^c$ structure of an almost complex structure J compatible with ω. Then $SW(\widetilde{P}_J) = 1$. There are many such J but they may give rise to isomorphic $Spin^c$ structures.

Here is a brief idea of how to prove the connected sum result.

Proof. Let Y and Z be as above and $X = Y\#Z$. Note that X has $b_2^+ \geq 2$ and so its SW invariant is independent of metric. Then there is a neck, diffeomorphic to $S^3 \times (0,1)$ which connects the two; it has positive scalar curvature. If we perturb the metric to lengthen this neck, we get that the SW invariant should vanish on the neck (because of its positive scalar curavture). It also turns out that if P_X is a $Spin^c$ structure on X, there is a way to construct it from $Spin^c$ structures P_Y and P_Z on Y and Z in some fashion. In fact, the $Spin^c$ structures have the following relationship: $\mathcal{S}_X \cong_{set} \mathcal{S}_Y \times \mathcal{S}_Z$.

Anyways, we have $\mathcal{M}(P_X) = \mathcal{M}(P_Y) \times \mathcal{M}(P_Z)$. On the other hand, let's compute the dimension of $\mathcal{M}(P_X)$. $c_l(L_X)^2 = c_1(L_Y)^2 + c_1(L_Z)^2$, $\sigma(X) = \sigma(Y) + \sigma(Z)$, and $\chi(X) = \chi(Y) + \chi(Z) - 2$. Thus, $\dim \mathcal{M}(P_X) = \dim \mathcal{M}(P_Y) + \mathcal{M}(P_Z) + 1$. Keep in mind that the virtual dimensions could be negative. For simplicity, suppose the dimension of $\mathcal{M}(P_X)$ is 0. Then on the RHS, one of the dimensions must be negative and hence, the space is empty. Thus, $\mathcal{M}(P_X)$ is empty. If $\dim \mathcal{M}(P_X) > 0$, we need some more argument but it the result still holds. $\square$

6 Applications

Here are a few applications. First, note that if (M^4, ω) is a symplectic manifold, then $\omega \wedge \omega$ is a volume form; the Poincaré dual of this is that the self intersection $\omega^* \cdot \omega^* > 0$. Hence, $b_2^+ > 0$. It was shown that the SW invariants of symplectic manifolds are nontrivial. Thus, using the connected sum vanishing result, we see that symplectic manifolds cannot be a smooth connected sum of two manifolds both with $b_2^+ > 0$. Of course, by Freedman's work, M could be topologically a connect sum.

6.1 Blowups of $\mathbb{CP}^2$

For $0 \leq n \leq 8$, $\mathbb{C}P^2 \# n\overline{\mathbb{C}P}^2$ is obtained as blowups and these give rational complex surfaces. $\mathbb{P}^1 \times \mathbb{P}^1$ is also a rational surface. As it turns out, all of these admit a Hitchins metric which has positive scalar curvature. So the Seiberg-Witten invariants vanish. However, there are constructions of smooth manifolds which are homeomorphic to these blowups for $n = 6, 7, 8$ but have nonvanishing SW invariant. Hence, those are somewhat "exotic" as their smooth structures do not allow for positive scalar metrics.

Now, a compact complex surface is rational if and only if its field of meromorphic functions is isomorphic to $\mathbb{C}(u, v)$. Donaldson theory and SW theory can be used to show that if a complex surface is diffeomorphic to a rational surface, then it itself is rational.

6.2 K3 Blown-up

Consider the K3 surface (its unique in the smooth category but not algebraic category). It has only two $Spin^c$ structure $\pm P_J$ from the complex structure, which gives us a nontrivial moduli space. Note that $c_1(\det P) = 0$ and $\chi(K3) = 24, \sigma(K3) = -16$. Also, $SW_{K3}(\pm P_J) = +1$; both are $+1$ because $1 + b_2^+ - b_1 = 4$.

Now let $X = K3\#\overline{\mathbb{C}P}^2$ and $Y = 3\mathbb{C}P^2\#20\overline{\mathbb{C}P}^2$. These two manifolds both have indefinite intersection forms (so neither positive nor negative definite) and neither is spin (by Rohklin's theorem). Also, the intersection forms have the same rank, signature, and type. The type of an intersection form is even if $D \cdot D$ is even for all $D \in H^2$ and is odd otherwise. Here, we have the signature is $3 - 20 = -17$ for both and the type is odd for both.

From the work of Serre, Milnor, and Husemoller, the rank, signature, and type form a complete set of invariants for indefinite unimodular symmetric forms.

It is the work of M. Freedman that compact, simply connected topological 4-manifolds are in 1-1 correspondence with pairs $\{(Q, \alpha)\}$ where Q is the intersection form and α the Kirby-Siebenmann obstruction (mostly easily described as $\alpha(M) = 0$ if $M \times S^1$ is smoothable and is 1, if not). When Q is even, we further require that $\sigma(Q)/8 \equiv \alpha \pmod 2$. However, Q is odd for X and Y and of both are compact and simply connected. Thus, X and Y are **homeomorphic**. Note that Freedman's result shows that every unimodular, symmetric bilinear form is the intersection form of some **topological** 4-manifold.

However, SW_X is nontrivial because the Seiberg-Witten invariant on a K3 of the standard $Spin^c$ structure coming from the complex structure is $+1$ and X has basic classes coming from E (arises from $\overline{\mathbb{C}P}^2$). Thus, $SW_X \neq 0$. However, $SW_Y \equiv 0$ because we can apply property (6) and (7) to Y. Thus, X and Y are **not diffeomorphic**. This fact was established by Donaldson using different methods but it's nice to confirm it with Seiberg-Witten theory.

Important point: Having this example proves that the Seiberg-Witten invariant can detect different smooth structures in some cases. But of course, it does so in a *negative* way: if two smooth manifolds have different SW invariants, then they are not diffeomorphic. But there are examples of 4-manifolds with the same SW invariants that are **not** diffeomorphic; seen by other means (e.g. Yamabe invariant). Of course, SW is also unable to do anything for us with, say S^4, which admits a positive scalar metric.

By the way, recall that every closed, orientable 4-manifold is $Spin^c$ and also cobordant to a connected sum of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P}^2$. Donaldson's Diagonalization establish that if M is a closed, simply connected **smooth** 4-manifold with definite intersection form, the intersection form is diagonalizable over $\mathbb{Z}$. This means that if Q is a non-diagonalizable unimodular, symmetric, bilinear form, though there is a **topological** 4-manifold M with Q as its intersection form, M does not admit any smooth structures.

6.3 More Kähler Surfaces

On a Kähler surface X, the canonical bundle $\Omega_X^{2,0}$ has 1st Chern class $c_1(X)$. We call it **positive** if there exists a Kähler form $\omega \in \Omega^{1,1}_{X,\mathbb{R}}$ such that $[\omega] = c_1(X)$. **Negative** if $[\omega] = -c_1(X)$, and **null** if $c_1(X) = 0$ in $H^2(X, \mathbb{C})$; so it could have torsion.

Yao showed that if X is a negative surface, then X admits a Kähler-Einstein metric g with negative Ricci curvature. Let λ be the scalar curvature.

Claim: On a negative surface with canonical $Spin^c$ structure coming from the complex structure, there exists a irreducible solution to the SW equations; it comes from the KE-metric g. Examples of such surfaces: degree $2k + 1$-hypersurfaces in $\mathbb{P}^3$ with $k \geq 2$. On the other hand, Freedman showed that such surfaces are topologically $n\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2$ with $m > 1$.

Furthermore, by Gromov-Lawson, a manifold of standard smooth type $n\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2$ with $m \geq 1$ has positive scalar curvature metrics and hence, no solutions to the SW equations. This shows that the negative surfaces of Yau must have **exotic** smooth structure.

6.4 Simple Type Conjecture

We end by stating a conjecture. Consider a closed, orientable smooth 4-manifold X. Let $\mathbb{B}_X$ denote the **support** of SW; that is, the set of $Spin^c$ structures on which SW does **not** vanish. Furthermore, assume now that $b_2^+(X) > 1$. X is said to be of SW**-simple type** if for every $P \in \mathbb{B}_X$, the dimension of the moduli space is zero. Contrapositively, if the moduli space has positive dimension, then SW vanishes on it. As of 2019, all the 4-manifolds with $b_2^+ > 1$ and computed SW invariants are of simple type.

Conjecture (Witten): All closed, orientable, smooth 4-manifolds with $b_2^+ > 1$ are of simple type.