Notes on M. Atiyah's *New Invariants of 3- and 4-Dimensional Manifolds*

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These are some notes I took to try and flesh out some of the ideas Atiyah mentions in this paper from a mathematical proceeding in honor of Hermann Weyl.

1 Introduction

One major idea in studying 3- and 4-manifolds is to "cut them in half." For example, we might take a self-indexing Morse function $f : Y \to [0, 3]$ on a closed, oriented 3-manifold and thus, obtain a Heegaard splitting. We then do Heegaard Floer theory on the surface $f^{-1}(3/2)$. Similarly, many 4-manifolds do not decompose as a connected sum by gluing along an S^3 but do decompose as a pseudo-connected sum along a homology 3-sphere.

2 Casson Invariant

Let Y be an oriented homology 3-sphere. This means $H_1(Y) = 0$ but of course, $\pi_1(Y)$ could be nontrivial. As we know now, if $\pi_1(Y) = 0$, then $Y \cong S^3$. The Casson invariant $\lambda(Y)$ is roughly defined as half the number of irreducible representations $\pi_1(Y) \to SU(2)$. What is the meaning of this if $SU(2)$ is not a vector space? The meaning is that we're looking at dim 2 complex irreducible representations $\rho : \pi_1(Y) \to \mathrm{Aut}(\mathbb{C}^2)$ but we're asking for $\mathrm{im}\,\rho \subset SU(2)$. We'll count conjugate representations as being the same so really, we mod this set out by $SU(2)$ under conjugation.

2.1 Irreducible Representations

Now, $H_1(Y)$ is the abelianization of $\pi_1(Y)$ and is zero. This means $\pi_1(Y) = [\pi_1(Y), \pi_1(Y)]]$. Atiyah claims that the only reducible representation in $SU(2)$ is the trivial one. How shall we understand this? Recall that a representation is irreducible if it doesn't have any sub-representation. So suppose that there were a sub-representation; i.e. $\pi_1(Y)$ acts on some $V \subset \mathbb{C}^2$ invariantly and $\dim_{\mathbb{C}} V = 1$. This splits $\mathbb{C}^2 = V \oplus W$. The only actions here is mutiplication by scalars. However, $\pi_1 = [\pi_1, \pi_1]$. Take an element of the form $[a, b] = a^{-1}b^{-1}ab$; since multiplication by scalars is commutative, then in fact, $[a, b]$ acts trivially by 1. Then all the commutators act trivially and so $\pi_1(Y)$ acts on V trivially. And if we quotient out by V, we have π_1 acting on $W = \mathbb{C}^2/V$. But this action then is also trivial. Therefore, if a representation is reducible, it is in fact, trivial. Hence, he says, "the only reducible representation in $SU(2)$ is the trivial one, so irreducible = nontrivial."

2.2 Flat Connections

How does one count such representations? Taubes showed that we can identify these representations with **flat connections** on the trivial bundle $Y \times SU(2)$. Recall that a connection is flat if and only if the holonomy depends only on the homotopy classes of loops.

Let $\mathcal{A}$ be the space of $SU(2)$-connections for the trivial bundle over Y and $\mathcal{G}$ the group of gauge transformation; that's just maps $Y \to SU(2)$ or, since the bundle is trivial, sections of the trivial bundle. We'll let $\mathcal{C} = \mathcal{A}/\mathcal{G}$. A reducible connection is one that has nontrivial stabilizer under gauge transformations. The space $\mathcal{C}$ is, away from singularities formed by reducible connections, an infinite dimensional manifold.

2.3 Constructing a Closed 1-form F

Atiyah says that we can define a natural 1-form F on $\mathcal{C}$. A 1-form, restricted to a point, should take in vectors. Somehow, we're able to identify tangent vectors on $\mathcal{A}$ with $\mathfrak{su}(2)$ valued 1-forms on Y. Then, for $A \in \mathcal{A}$, we can take it's curvature 2-form F_A which is also $\mathfrak{su}(2)$ valued. So take $\omega \in T_A\mathcal{A}$; I think we define $F(\omega) = \int_Y \omega \wedge F_A$. Moreover, the claim is that the Bianchi identity, which says $d_A F_A = 0$, says that F is $\mathcal{G}$ invariant and so descends to $\mathcal{C}$.

The zeros of F are the flat connections and so a count of irreducible representations $\pi_1(Y) \to SU(2)$ is a count of the zeros of F. It seems that the flat connections, modulo gauge transformations is finite and so countable but we need to be careful about the reducible connections. Hence, Fredholm perturbation is necessary.

With regards to this statement about representations of groups whose abelianization is zero. In general, I think there is little we can say but here, we are looking specifically at 2 dim representations. My initial thought had been, if G is an abelian group, then all its irreducible representations are 1-dim. So any larger representation will always contain a subrepresentation. The abelianization, in some sense, measures how abelian a group is. If the abelianization is the group itself, then the group is abelian to start with. If the abelianization is small, then there are many non-commuting elements. My thought was: "Highly abelian groups give us smaller irreps while highly non-abelian groups can give many irreps which have larger dimension." I don't know if this is a good guiding principle. Probably not, except when we restrict our attention to 2 dim irreps.

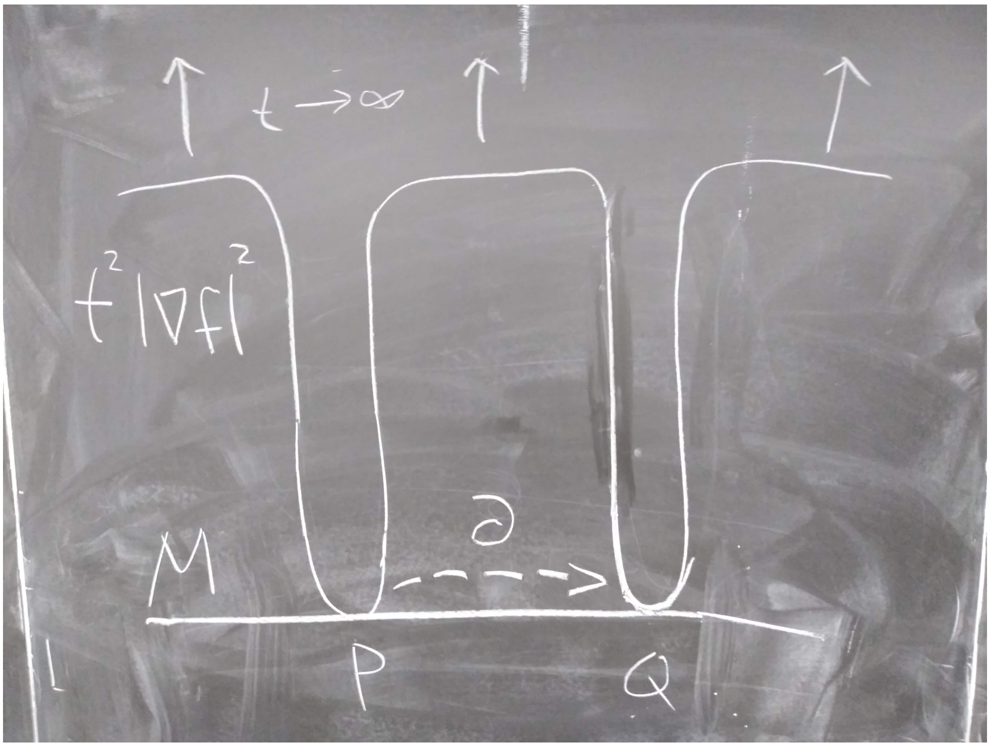
3 Morse Theory

3.1 Witten's Interpretation

Witten has an interesting take on Morse theory that I'll like to understand. Let M be a compact manifold M with Morse function f. We'll think of homology here as Hodge-de Rham homology, represented by harmonic forms. But we can modify our Laplacian by f and real parameter t. Let's replace d with $d_t := e^{-tf}d\, e^{tf}$ and d^* with $d_t^* := e^{tf}d^*\, e^{-tf}$. So multiply by some function, then apply d or d^*. Let $\Delta_t = d_t d_t^* + d_t^* d_t$. With some computations, we can show that

$$\Delta_t = \Delta + t^2|\nabla f|^2 + \sum_{i,j} \frac{\partial^2 f}{\partial \phi^i \partial \phi^j}[a^{*i}, a^j].$$

Forgetting about the meaning of the last term, just consider a zero eigenform ("harmonic" forms for the deformed Laplacian) α of Δ_t; i.e. $\Delta_t \alpha = 0$. If t is large, note that in order for $t^2|\nabla f|^2\alpha$ to not be too big, we need α to vanish away from the critical points of f since, it is only at the critical points of f is $\nabla f = 0$. Put another way, α should be concentrated near the critical points of f. In the limit, $t \to \infty$, α is sort of like a Dirac delta-form at each of the critical points. These types of eigenforms are called **classical ground states**. The picture I have in mind of the potential $t^2|\nabla f|^2 \to \infty$ is a potential energy well where the potential gets larger and larger.



Potential Energy Well

Let P, Q be two critical points of f with the index of Q being one less than that of P. Atiyah says that the eigenform associated to a critical point Q of f has an exponentially small correction due to Q which can approximately be computed by using the trajectories of ∇f from P to Q. This is called **quantum mechanical tunneling**, which describes the **probability** of the transition P to Q. The point is that a ground state is sort of stable and to leave the potential energy well requires too much energy. But quantum mechanically, there's a probability of moving to another ground state. Thus, this is Witten's QM interpretation of the differential ∂ in the Morse chain complex.

3.2 Hydrogen Fusion

By the way, a classic example of quantum tunneling is in the sun. The surface temperature of the sun is 5,778 K and its core temperature is 15.7 million K. But classically, the temperatures needed for hydrogen fusion to occur is 100 million K, far higher. So how can the sun undergo hydrogen fusion? The answer is quantum tunneling. I think the idea is that energy and time are "sort of dual" to each other, similar to how position and momentum are dual in Heisenberg's Uncertainty Principle. Anyways, there is some probability of a pair of nucleons being in some other stable state, such as being bound as deuterium instead of separated particles. So though it doesn't have enough energy to classically cross the energy barrier, it "tunnels" through the barrier.

From Wikipedia: "The proton–proton chain occurs around 9.2×10^{37} times each second in the core, converting about 3.7×10^{38} protons into alpha particles every second (out of a total of $\sim 8.9 \times 10^{56}$ free protons in the Sun), or about 6.2×10^{11} kg/s. Fusing four free protons into a single alpha particle releases around 0.7% of the fused mass as energy, so the Sun releases energy at the mass–energy conversion rate of 4.26 million metric tons per second."

4 Floer Homology

4.1 The Chern-Simons Functional

The 1-form we saw in section 1 is in fact closed. Thus, locally, it is the differential of a well-known function on $\mathcal{C}$: the Chern-Simons functional. Let A_0 be the trivial connection on the trivial bundle $Y \times SU(2)$ and let $A_t = (1-t)A + tA_0$, for $t \in [0,1]$. A_t can be thought of as a connection on the 4-manifold $Y \times I$.

$$f(A) = \frac{1}{8\pi^2} \int_{Y \times I} \operatorname{tr} |F_{A_t}|^2 \, vol = \frac{1}{8\pi^2} \int_{Y \times I} \operatorname{tr} F_{A_t} \wedge *F_{A_t},$$

where F_{A_t} is the curvature of A_t on $Y \times I$. We will soon just drop the t's. If we were looking at this integral but on a closed 4-manifold M, the integral will always give us the 2nd Chern class of M which is also the Euler class. This is a rather remarkable result. It says that for closed 4-manifolds, we may obtain topological information when our starting point is geometric information.

In our situation, $f(A)$ is invariant under action by $\mathcal{G}_0 \subset \mathcal{G}$, the identity component. Since $\mathcal{G}/\mathcal{G}_0 = \mathbb{Z}$, $f(A)$ changes by integers under the full group. To avoid this issue, f is well-defined on $\mathbb{R}/\mathbb{Z}$. Why is it that $\mathcal{G}/\mathcal{G}_0 = \mathbb{Z}$? Since Y and $SU(2)$ are both connected, compact, oriented 3-manifolds, then we can discuss the degree of a map. Degree is a homotopy invariant. If I am not mistaken, since $\mathcal{A}$ can be modeled on an affine space and homotopy classes of elements of $\mathcal{G}$ are determined by degree, then $\pi_1(\mathcal{C}) = \mathbb{Z}$.

Now, we may try applying Morse theory to this setting but we'll run into two main problems:

1. f takes values in $\mathbb{R}/\mathbb{Z}$ instead of $\mathbb{R}$.

2. The Hessian of f at a critical point will give a bilinear form whose positive and negative eigenspaces are both infinite dimensional.

The first problem can be circumvented if we take a covering space $\mathcal{C}_0 = \mathcal{A}/\mathcal{G}_0$. Recall that the identity component of a topological group is always a closed normal topological subgroup.

4.2 Spectral Flow

To understand the second issue, let us take a detour into a different infinite dimensional setting. One setting of Morse theory is that of geodesics of a Riemannian manifold. The Hessian in this setting, if we view it as an operator rather than a quadratic form, is of **Laplace type**: a second order elliptic operator. Hence, it's negative eigenvalues are bounded below and so n^-, the dimension of the negative eigenspace, is finite. Thus, Morse theory can be done even in this infinite dimensional setting. However, when we look at the setting of flat connections, we find that the Hessian is of **Dirac type**: it is of first order and we have no such lower bounds on the negative eigenvalues.

The trick is to try and define some **relative** Morse index of critical points P and Q which, intuitively, should be the dimension of the unstable manifold of P intersected with the stable manifold of Q. Of course, the hope is that the dimension is finite. We would then denote this quantity as $n^-_{P,Q}$. Let's proceed in steps.

1. Fix a metric on Y and extend the Hessians at critical points to a continuous family of self-adjoint Dirac type operators; this is parametrized by $C \in \mathcal{C}$.

2. Then, we may consider a path of operators H_t between H_P and H_Q by considering a path between critical points P, Q of f. We count the net number of negative eigenvalues of H_P that cross over and end up as a positive eigenvalue of H_Q. This integer is called the **spectral flow** and only depends on the homotopy class of the path from P to Q.

3. The spectral flow would be well-defined if $\mathcal{C}$ were simply-connected. Since it is not, we consider the spectral flow lifted to $\mathcal{C}_0$. This step involves seeing that $n^-_{P,Q}$ is defined modulo the spectral flow around a generating closed loop in $\mathcal{C}$. Above, I argue why $\pi_1(\mathcal{C}) = \mathbb{Z}$.

4. The spectral flow around a closed loop can be computed from the index theorem of Atiyah and Singer on $Y \times S^1$. We find that the answer is 8. Thus, this will give us a $\mathbb{Z}_8$ grading.

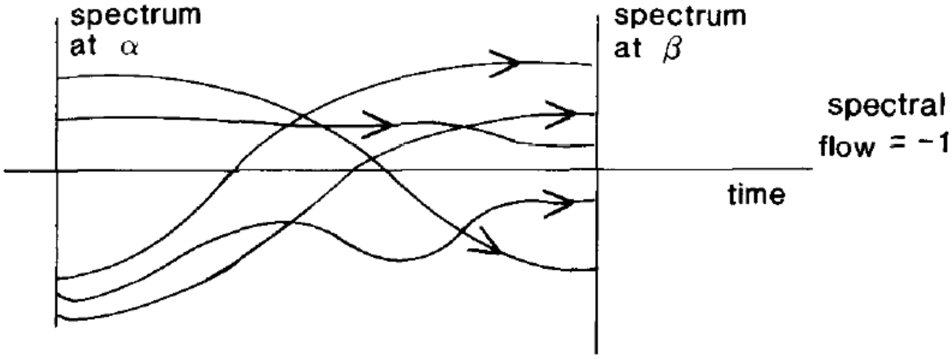


Image from *Floer Homology Groups for Homology Three-Spheres* by Peter Braam

4.3 The Chain Complex

We now assume that everything is nondegenerate or that some perturbation *à la* Taubes gives us nondegeneracy. We proceed by defining chain groups C_* indexed modulo 8. The differential ∂ will be defined via a count of trajectories between critical points P and Q, whose indices differ by one. Lastly, we'll want to prove that the homology groups are independent of perturbations and such. Before giving more detail about defining ∂, note that there is no obvious choice for H_0; we see the English meaning for saying $\mathbb{Z}_8$ is cyclic; there's not starting point! However, if we're careful and mark the trivial representation, then there is a way to identify H_0.

Now to defining ∂. As expected, we would like to count trajectories of ∇f. What exactly does this mean? The gradient is defined like so: $(df)_A(X) = \int X \wedge *F_A = (X, \nabla f)_{L^2} = \int \langle X, \nabla f \rangle \, vol$. Therefore, $\nabla f = *F_A$. A trajectory of ∇f should then satisfy the equation

$$\frac{dA}{ds} = - * F_A.$$

This equation, interpreted on the infinite cylinder $Y \times \mathbb{R}$ is precisely the anti-self dual equation which defines instantons. The boundary conditions were impose here are that as $s \to \pm\infty$, the flat connections converge to P and Q, respectively. Taking Witten's interpretation of ∂ as a tunneling effect, we are using instantons to tunnel from the ground state (or vacuum) of one flat connection to the ground state of another. This is precisely the way physicists use instantons and was their original motivation. Witten even uses the word "instanton" in his paper *Supersymmetry and Morse Theory.* After quotienting by translations, we should obtain a zero dimensional moduli space of trajectories and thus, be able to count.

Of course, there are technicalities to be wary of. Counting the trajectories with sign is one such caveat. The modulo 8 may be viewed as a refinement of the modulo 2 labeling that determines the sign of the Casson invariant. But even more pertinent are questions about the compactness of the moduli spaces of trajectories and what happens to compactness as we perturb. And also how to show $\partial^2 = 0$. The work of Donaldson, Taubes, and Uhlenbeck tells us that things work out. Thus, we have a new invariant for homology 3-spheres in the form of homology groups $HF_q(Y)$, $q \in \mathbb{Z}_8$.

Just as in the finite dimensional case where taking $-f$ instead of f gives Poincaré duality, this occurs here as well. Depending on the orientation of Y, we have dual homology groups $HF^+(Y)$ and $HF^-(Y)$.

Moreover, twice the Casson invariant is the Euler characteristic:

$$2\lambda(Y) = \sum_{q=0}^{7} (-1)^q \dim HF_q^+(Y).$$

5 Relation with 2 Dimensions

We can, of course, relate this to 2 dimensions if we take a Heegaard splitting; use a self-indexing Morse function on Y^3. Let X be the smooth surface that we obtain with genus g and Y^+, Y^- be the g-handlebodies with X as boundary. $\pi_1(X)$ is generated by $A_1, ..., A_g, B_1, ..., B_g$ and, for example, the map $\pi_1(X) \to \pi_1(Y^+)$ sends all the $B_i \to 1$ and the images of A_i freely generate $\pi_1(Y^+)$. A similar story holds for $\pi_1(Y^-)$.

Casson's idea for defining $\lambda(Y)$ was to use the following diagram.

$$\begin{array}{ccc} \pi_1(X) & \longrightarrow & \pi_1(Y^+) \\ \downarrow & & \downarrow \\ \pi_1(Y^-) & \longrightarrow & \pi_1(Y) \end{array}$$

All the maps are induced by inclusion. Then, a representation $\pi_1(Y) \to SU(2)$ is the same as a pair of representations of $\pi_1(Y^{\pm})$ which agree when pulled-back on $\pi_1(X)$. These classes of representations form a moduli space M which, after removing the reducible representations, is a $6g - 6$ (real) dimensional Kähler manifold. We can define a symplectic form on this M, defined on the generators A_i, B_k's.

I believe we need to assume $g \geq 2$. The representations of X which extend to $Y^{\pm}$ give Lagrangian submanifolds $L^{\pm} \subset M$ of dimension $3g - 3$. We can define the Casson invariant by $2\lambda(Y) = L^+ \cap L^-$. **Question:** Is this true? Or do we need Lagrangian Floer homology $HF_*(L^+, L^-)$? I think the Atiyah-Floer conjecture is for a general 3-manifold Y, not necessarily a homology sphere, the Lagrangian Floer homology $HF_*(L^+, L^-)$ should be isomorphic to the instanton Floer homology we defined previously. The theory probably works when $H_1(Y) \neq 0$ and homology 3-spheres may be the examples that motivated the formulation of this conjecture.

Another questions the grading on Lagrangian Floer theory is modulo $2N$ where N is the minimal Chern number while for a homology 3-sphere, the grading is mod 8. So is the minimal Chern number $N = 4$ for whatever the surface X in Y turns out to be?

Geometrically, a path on M is a 1-parameter family of flat connections on X. This can be viewed as a single connection on $X \times \mathbb{R}$. The boundary conditions coming from our considerations of $L^{\pm}$, say that the connection should extend over $Y^{\pm}$ as $t \to \pm\infty$. Thus, the symplectic theory for paths in M should be relaed to a limiting case of the Floer theory for the space $\mathcal{C}$ of connections on Y. It's sort of like stretching Y out where there ends are $Y^{\pm}$ and are connected by $X \times \mathbb{R}$.

6 Donaldson Invariants

6.1 4-Manifold Theory since Gauge Theory

Our setting now is that of an oriented, simply connected, closed smooth 4-manifold Z; b_2^+ and b_2^- will be the $\pm$ terms in the diagonalization of the intersection form. We'll also assume $b_2^+ > 1$

is odd. If Z is a complex algebraic surface, a theorem of Hodge tells us that $b_2^+ = 1 + 2p_g$ where p_g is the geometric genus. Since we're assuming that Z is nonsingular, we take $p_g = h^{2,0}$; the dimension of closed holomorphic 2-forms. In general, $p_g = h^{n,0}$; a quantity to describe how many holomorphic volume forms we have. So all we need is for $p_g > 0$ for Z to be in our setting.

The Donaldson invariants are a sequence of integer polynomials ϕ_k on $H_2(Z)$ where $k > k_0$ is sufficiently large and the degree of ϕ_k is $d(k) = 4k - 3(b_2^+ + 1)/2$. Here are two important results of Donaldson.

Theorem 6.1. *If $Z = Z_1 \# Z_2$ is a connected sum with $b_2^+(Z_i) \neq 0$ for $i = 1, 2$, then $\phi_k(Z) \equiv 0$ for all k.*

Theorem 6.2. *If Z is algebraic, then for $k > k_1(Z)$ (so for k larger than some constant depending on Z), $\phi_k(Z) \not\equiv 0$*

These two theorems in conjunction show that if we view an algebraic surface as just a smooth manifold, it is indecomposable. Of course we always have blowups, say $Bl_0\mathbb{P}^2$, but that decomposes into $\mathbb{P}^2 \# \overline{\mathbb{P}}^2$ and $b_2^+(\overline{\mathbb{P}}^2) = 0$.

In general, it is hard to compute these invariants which arise from studying instantons. However, for algebraic surfaces, Donaldson showed that the moduli space of bundles with anti self-dual connection and the moduli space of stable holomorphic bundles effectively coincide. From there, we really just need to study stable bundles.

For a nonalgebraic 4-manifold Z, suppose its quadratic form Q decomposes as a direct sum $Q = Q_1 \oplus Q_2$ with $b_2^+(Q_i) \neq 0$ for $i = 1, 2$. If $\phi_k(Z) \not\equiv 0$ for all k, then Z cannot be decomposed smoothly. Interestingly, M. Freedman showed that there exist topological manifolds Z_1, Z_2 with intersection forms Q_1, Q_2 such that $Z = Z_1 \# Z_2$ as a topological manifold. Of course, the connected sum has a neck which looks like $S^3 \times I$. However, when it comes to these indecomposable 4-manifolds, it is known that instead of decomposing along S^3, we can decompose along some homology 3-sphere Y which induces the algebraic decomposition $Q = Q_1 \oplus Q_2$. This shows that the indecomposability of Z is somehow reflected in the nontriviality of the homology 3-sphere Y.

Let us now move towards seeing how Donaldson related his invariants on Z to Floer homology groups on Y. We begin by recalling the definition of Donaldson invariants ϕ_k. We fix a Riemannian metrix Z, a positive integer k, and consider the space $M_k(Z)$ of k-instantons on Z. A k-instanton is a connection A whose curvature F_A satisfies $*F_A = -F_A$ and satisfies

$$\frac{1}{8\pi^2} \int \operatorname{tr} F_A \wedge *F_A = k.$$

6.2 Detour: the 2nd Chern Class

Let me make a detour here. Chern-Weil theory tells us that the integral above is in fact, the 2nd Chern class of the principal $SU(2)$ bundle. This is remarkable since, once we fix a $SU(2)$ bundle $P \to Z$, all of its connections, when we integrate their curvature as in the above expression, give the **same** constant. Moreover, c_2 completely classifies $SU(2)$ bundles over a closed, oriented 4-manifold. To see why, recall that isomorphism classes of $SU(2)$ bundles are in 1-1 correspondence with homotopy classes of maps $Z \to BSU(2)$. Using the long exact sequence of homotopy groups for the fibration $G \to EG \to BG$ and also using the fact that EG is contractible, we can show that $\pi_{k+1}(BG) = \pi_k(G)$. But $G = SU(2) \cong S^3$ and so π_1, π_2, π_3 of $BSU(2)$ are all trivial and $\pi_4(BSU(2)) = \mathbb{Z}$. For anything higher, it doesn't really matter since we're looking at 4-manifolds. So effectively, $BSU(2)$ behaves like the Eilenberg-MacLane space $K(\mathbb{Z}, 4)$ for 4-manifolds. On the other hand, the homotopy classes of maps $Z \to K(\mathbb{Z}, 4)$ is identified with $H^4(Z, \mathbb{Z}) = \mathbb{Z}$. There is a **universal second Chern class**

$c_2 : BSU(2) \to K(\mathbb{Z}, 4)$; and a classifying map $f : Z \to BSU(2)$ can now be thought of as an element of $H^4(Z, \mathbb{Z})$ via c_2. The upshot is that c_2 completely classifies all principal $SU(2)$ bundles over a 4-manifold Z.

6.3 The Donaldson Polynomials

Let's return to instantons. $M_k(Z)$ is a manifold for generic metrics and has dimension $2d(k)$. If $d(k) = 0$, then we can count them and treat that as a Donaldson invariant (how do we have compactness?). If I'm not mistaken, when $k = 1$, this is the setting in which Donaldson proved the diagonalization theorem. For $d(k) > 0$, fix $d(k)$ spherical cycles $\alpha_i : S^2 \to Z$ with homology classes $[\alpha_i]$. Each cycle defines a codimension 2 submanifold A_i of M_k (not clear how). The submanifold consists of connections which, when pulled back to S^2 via α_i, are special connections. This just means they define nontrivial holomorphic bundles. I think the point is that in complex dim 2, a unitary connection defines a holomorphic structure.

Now, consider the intersection number $A_1 \cap ... \cap A_d$ as a function of $\alpha_1, ..., \alpha_d$. It depends only on the homology classes and its values define $\phi_k([\alpha_1], ..., [\alpha_d])$ as a symmetric d-linear function on $H_2(Z)$. M_k is **not** compact so we have to be careful about what happens at ∞. Somehow, this is why we need the condition that $k > k_0$. The idea is that we can compactify M_k but we want to do so by adding in some space of codim ≥ 2 so that we can define, with no further trouble, intersection numbers via the fundamental class $[M_k]$.

7 Relation to Floer Homology

Since we can have a decomposition of Z along a homology 3-sphere Y, we try studying the instanton equations on $Z^\pm$ and then matching on the boundary Y.

7.1 A Prototype

Consider an analogous prototype. Cutting S^2 along the equator, we can consider holomorphic functions on the hemispheres and compare their boundary values. The space of holomorphic functions on a disk is called the Hardy space. So we'll have two of them: $H^\pm$ and $H^+ \cap H^-$ is just the constant functions since the only holomorphic functions that agree on the boundary equator would have to be constant.

Let's change the prototype to be nonlinear now. We look at holomorphic maps $S^2 \to P$ where P is a complex manifold, say projective space. When we cut S^2, a holomorphic map in either hemisphere is determined by its restriction to S^1 which is a then a map $S^1 \to P$; so the holomorphic map is an element of the free loop space LP. Thus, we get $H^\pm$ subspaces of LP but they are not linear. Still, the global holomorphic maps are given by $H^+ \cap H^-$ and this should be finite dimensional. Linearizing the problem, we can recover the holomorphic maps $S^2 \to P$. Atiyah says that it is for this reason, we may think of $H^\pm$ as ∞ dimensional manifolds of approximately "half" the dimension of LP because intersecting them gives a finite dimensional subspace.

7.2 Lagrangian Floer Approach to ϕ_k

Let us return to the 4-dim situation. The instanton equations are 1st order so a solution is determined by the appropriate boundary data. Here, that's just a connection on Y, up to gauge equivalence. Thus, we should look at $\Sigma^\pm \subset \mathcal{C}(Y)$, the spaces consisting of boundary values of solutions to instanton equations in $Z^\pm$. The intersection $\Sigma^+ \cap \Sigma^-$ give global solutions on Z.

For simplicity, suppose $d(k) = 0$. The Donaldson invariant is an integer in this case and describes the algebraic number of k-instantons on Z. This number should be computed as the intersection number of Σ^+ and Σ^-. This requires some kind of homology theory for $\mathcal{C}(Y)$ in which $\Sigma^\pm$ are cycles that we can intersect. The Floer homology groups $HF^\pm(Y)$ provide this framework that we seek.

Let's draw on a picture from Morse theory. Suppose we have a geometric cycle α and we want to associate to it a cycle in the Morse complex. We can push it along the gradient flow and see which critical points it gets stuck on; i.e. the gradient flow doesn't push some points of α any further, after it "hangs" on the critical point. If β is a cycle of complementary dimension, we can deform β along the **ascending** gradient flow of f and see which critical points it gets stuck on. Then the intersection number $\alpha \cdot \beta$ can be reduced to local calculations near the critical points, in Morse charts.

For the ∞ dimensional manifold $\mathcal{C}$, the gradient flows are only defined for appropriately restricted initial data because we're solving a heat equation for an operator which is not bounded below nor above. However, we may use cycles $\Sigma^\pm$. They provide the right data for the two opposite flows. This is what Donaldson does: assign classes $[\Sigma^+] \in HF^+(Y)$ and $[\Sigma^-] \in HF^-(Y)$. Their pairing, using the Poincaré duality map $HF^+(Y) \otimes HF^-(Y) \to \mathbb{Z}$ gives the Donaldson invariant $\phi_k(Z)$.

This was all done under the assumption of $d(k) = 0$. When $d(k) > 0$, we can do something similar. Given classes $[\alpha_i]\ in H_2(Z)$ for $i = 1, ..., d$, choose an integer $r \in [0, d]$ and representative spherical cycles $\alpha_i : S^2 \to Z^+$, $i = 1, ..., r$ and $\alpha_j : S^2 \to Z^-$, $j = r + 1, ..., d$.

The boundary values of instantons on Z^+ which are special on $\alpha_1, ..., \alpha_r$ (pullback to define nontrivial holomorphic bundles of S^2) define some kind of a "cycle" $\Sigma^+(\alpha_1, ..., \alpha_r) \subset \mathcal{C}$, and hence a Floer homology class $[\Sigma^+(\alpha_1, ..., \alpha_r)] \in HF^+(Y)$. Similarly, define $[\Sigma^-(\alpha_{r+1}, ..., \alpha_d)] \in HF^-(Y)$. Pairing these defines a morphism of symmetric products

$$S^r(H_2(Y^+)) \otimes S^{d-r}(H_2(Z^-)) \to \mathbb{Z}.$$

Since $H_2(Z) = H_2(Z^+) \oplus H_2(Z^-)$, then

$$S^d(Z) = \bigoplus_{r=0}^{d} S^r(H_2(Z^+)) \otimes S^{d-r}(H_2(Z^-)).$$

By summing over r, we have a morphism $S^d(H_2(Z)) \to \mathbb{Z}$. Donaldson shows that this is in fact, his invariant ϕ_k. So we have a way to think of Donaldson invariants in a Floer theoretic way.

It seems that if we have a manifold Z^+ with boundary being a homology 3-sphere, there is a sequence of polynomials ϕ_r on $H_2(Z^+$ with values in $HF^+(Y)$. Then, we can get some kind of a relative Donaldson polynomial on this manifold with boundary.

Also, if Y is the standard 3-shere so that Z is the usual connected sum, then $HF^+(Y) = 0$ and Theorem 6.1 follows pretty quickly from computing $\phi_k(Z)$ using this Floer theoretic picture. Conversely, if we know that Z is indecomposable, then it follows that $HF^+(Y) \neq 0$. So Theorem 6.2 implies the nontriviality of Floer homology groups for many homology 3-spheres which come about in a pseudo-decomposition of algebraic surfaces. This is a nice way to show that there are homology 3-spheres not diffeomorphic to the standard S^3.

8 Concluding Remarks

8.1 Multiple Definitions for Singular Homology

We have various ways of defining homology groups depending on the choice of chain complex:

1. The Witten complex of a Morse function

2. The classical chain complex of geometrically defined cycles

3. de Rham complex

4. Hodge theory

Witten's ideas relate (1) to (4) while pushing cycles along the gradient flow relates (1) and (2). (3) and (4) are related by elliptic theory. Floer applied ideas of (1) to ∞ dimensional manifolds by treating the Chern-Simons functional and other action functionals like Morse functions. Donaldson's work is more like (2), using cycles $\Sigma^{\pm}$.

8.2 Middle Dimensions

Atiyah spends some time to make this idea of "middle dimension" more rigorous. Consider a product space of two ∞ dimensional spaces: $S^+ \times S^-$. We can define ordinary homology by taking finite-dim cycles in both factors or ordinary cohomology by taking finite-codim cycles in both factors. But there are also cycles of the type (cofinite)$\times$(finite) or (finite)$\times$(cofinite). This gives a sense of middle dimension homology, the first is "positive" and the other "negative."

But since our spaces, such as $\mathcal{C}$, are generally not products (only locally), the middle-dimensional homology cannot reduce to ordinary homology or cohomology by factorization. This means that Floer homology groups are, from a topological point of view, something **essentially new.**

8.3 ∞ Dimensional de Rham Theory and QFT

Suppose we have an infinite orthonormal basis e_n, $n \in \mathbb{Z}$, for some space T_C given by the spectrum of H_C so that T_C^+ is spanned by $n \geq 0$. The "volume element" of T_C^+ is $\omega = e_0 \wedge e_1 \wedge ...$ We also have infinite wedge products which differ from ω in only finitely many terms. Dualizing and taking linear combinations should define the "positive" differential forms Ω^+ at C. We can do something similar for defining Ω^-

The semi-infinite volume element ω is familiar in physics as the vacuum vector of a Fermionic Fock space. Ω^+ are the fields of a Fermionic quantum field theory. Let Δ_f^+ be a Laplacian where f is the Chern-Simons functional. This should be the Hamiltonian of the QFT and the harmonic forms are the ground states. Purely formally then, and ignoring the ground state, the Floer homology groups $HF+(Y)$ are the ground states of the QFT with Hamiltonian Δ_f^+. Atiyah says this discussion is found implicitly in Witten's *Supersymmetry and Morse Theory.* Witten knows about QFT as middle-dimensional homology.

Apparently this discussion is not rigorous for $3+1$ QFT, especially as there is no rigorous definition for Hamiltonians. But Floer's symplectic theory for paths in a symplectic manifold is a $1+1$ QFT and that's in better shape and has been investigated in string theory.

8.4 Open Problems (as of 1988)

1. The Atiyah-Floer conjecture: show that instanton Floer homology coincides with Lagrangian Floer theory done on a Heegaard splitting. This would appear as a comparison of $3+1$ and $1+1$ QFTs.

2. Produce an algorithm for computing $HF(Y)$ which generalizes Casson's algorithm.

3. Find a way to compute Donaldson invariants $S^*(H_2(Z^+)) \to HF^+(Y)$ when $Y = \partial Z^+$.

4. Find a connection with the link invariants of Vaughan Jones.

To see why the last suggestion is reasonable, Atiyah lists some properties shared by Floer homology and the Jones polynomial.

1. Both are subtle 3-dim invariants
2. They are sensitive to orientation, unlike the Alexander polynomial.
3. They depend on Lie groups
4. There are 2-dim schemes for computing these 3-dim invariants.
5. The variable in the Alexander polynomial corresponds to $\pi_1(S^1)$ but the variable in the Jones polynomials appears to relate to $\pi_3(S^3)$.
6. Both have deep connections with physics, namely QFT and statistical mechanics.