The Action Functional and its Differential

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The aim of this document is to compute the differential of the action functional. Let (W, ω) be a closed symplectic manifold. We assume that $\pi_2(\omega) = 0$, that is, for any smooth map $f : S^2 \to W$,

$$\int_{S^2} f^*\omega = 0.$$

We also consider a time-dependent Hamiltonian $H : W \times \mathbb{R} \to \mathbb{R}$. We may assume H to be periodic in t. The Hamiltonian system is the following ODE: $\dot{x}(t) = X_t(x(t))$. Note its similarity to the defining ODE for the flows of a vector field. Let $\mathcal{L}W$ be the free loop space of contractible loops on W; that is, smooth contractible maps $x : S^1 \to W$. Since they are contractible, we may extend them to maps on the disk $u : D^2 \to W$.

The action functional is $\mathcal{A}_H : \mathcal{L}W \to \mathbb{R}$, defined as

$$\mathcal{A}_H(x) = -\int_{D^2} u^*\omega + \int_0^1 H_t(x(t))\, dt,$$

where u is an extension for x to $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$. The second integral is well-defined but the first depends on the choice of extension. However, if v is another extension, then we may glue u and v along their boundaries to form a map $f : S^2 \to W$ and

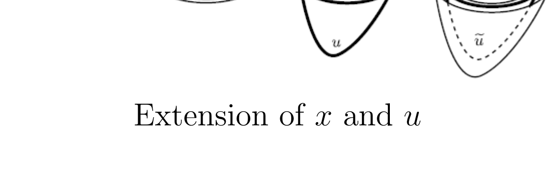
$$\int_{S^2} f^*\omega = \int_{D^2} u^*\omega - \int_{D^2} v^*\omega = 0$$

by the $\pi_2(\omega) = 0$ assumption. Thus, the functional is well-defined.

Proposition 0.1. *A loop x is a critical point of $\mathcal{A}_H$ if and only if $t \mapsto x(t)$ is a periodic solution to the Hamiltonian system $\dot{x}(t) = X_t(x(t))$*

Proof. We compute the differential of $\mathcal{A}_H$ at a point $x(t)$ and at a tangent vector $Y(t)$ (a section of the vector bundle x^*TW). We extend x to $\tilde{x}(s,t)$ in a neighborhood of of x: $\tilde{x} : (-\epsilon, \epsilon) \times S^1 \to W$ where $\tilde{x}(0,t) = x(t)$ and $\partial\tilde{x}/\partial s(0,t) = Y(t)$. This means that

$$(d\mathcal{A}_H)_x(Y) = \frac{\partial}{\partial s}\mathcal{A}_H(\tilde{x})|_{s=0}.$$



Extension of x and u

To make this computation, we need to extend $u : D^2 \to W$ as well. Let $\tilde{u}(z, z)$ be the extension and $\tilde{u}(0, z) = u(z)$ and $\tilde{u}(s, e^{2\pi i t}) = \tilde{x}(s, t)$. Extend Y by setting

$$Y(z) = \frac{\partial \tilde{u}}{\partial s}(0, z).$$

We now have

$$\mathcal{A}_H(\tilde{x}(s,t)) = -\int_{D^2} \tilde{u}^*\omega + \int_0^1 H_t(\tilde{x}(s,t))\, dt.$$

Differentiating the first term gives

$$\begin{aligned} -\int_{D^2} \frac{d}{ds}\tilde{u}^*\omega\bigg|_{s=0} &= -\int_{D^2} u^*(\mathcal{L}_{Y(z)}\omega) = -\int_{D^2} u^*(d\iota_{Y(z)}\omega) \\ &= -\int_{S^1} x^*(\iota_{Y(t)}\omega) = -\int_0^1 \omega(Y(t), \dot{x}(t))\, dt \\ &= \int_0^1 \omega(\dot{x}(t), Y(t))\, dt \end{aligned}$$

Differentiating the second term gives

$$\begin{aligned} \int_0^1 \frac{\partial}{\partial s} H_t(\tilde{x}(s,t))\bigg|_{s=0} dt &= \int_0^1 (dH_t)_{\tilde{x}(0,t)}(Y(t))\, dt \\ &= \int_0^1 \omega_{x(t)}(Y(t), X_t(x(t)))\, dt. \end{aligned}$$

Combining the two terms gives us

$$(d\mathcal{A}_H)_x(Y) = \int_0^1 \omega(\dot{x}(t) - X_t(x), Y)\, dt.$$

The nondegeneracy of ω says that this last line equals 0 if and only if $\dot{x} = X_t(x)$. $\square$